

Twisted $3D \mathcal{N} = 4$ Supersymmetric YM on deformed \mathbb{A}_3^* Lattice

El Hassan Saidi*

1. LPHE-Modeling and Simulations, Faculty Of Sciences, Rabat, Morocco
2. Centre of Physics and Mathematics, CPM- Morocco

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Abstract

We study a class of twisted $3D \mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory on particular 3- dimensional lattice denoted as $\mathcal{L}_{3D}^{su_3 \times u_1}$ and given by non trivial fibration $\mathcal{L}_{1D}^{u_1} \times \mathcal{L}_{2D}^{su_3}$ with base $\mathcal{L}_{2D}^{su_3} = \mathbb{A}_2^*$, the weight lattice of $SU(3)$. We first, develop the twisted $3D \mathcal{N} = 4$ SYM in continuum by using superspace method where the scalar supercharge Q is manifestly exhibited. Then, we show how to engineer the 3D lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$ that host this theory. After that we build the lattice action \mathcal{S}_{latt} invariant under the 3 following: (i) $U(N)$ gauge invariance, (ii) BRST symmetry, (iii) the hidden $SU(3) \times U(1)$ symmetry of $\mathcal{L}_{3D}^{su_3 \times u_1}$. Other features such as reduction to twisted $2D$ supersymmetry with 8 supercharges living on $\mathcal{L}_{2D}^{su_2 \times u_1}$, the extension to twisted maximal $5D$ SYM with 16 supercharges on lattice $\mathcal{L}_{5D}^{su_4 \times u_1}$ as well as the relation with known results are also given.

Keywords: Reduction of chiral $6D \mathcal{N} = (1, 0)$ SYM, BRST symmetry and Scalar supersymmetry, Twisted SYM on lattice, Root and weight lattices of $SU(k)$.

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*h-saidi@fsr.ac.ma

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1 Introduction

Following [1]-[9] and refs therein, the lattice version of maximal euclidian¹ four dimensional $\mathcal{N} = 4$ supersymmetric Yang Mills theory (SYM) with $U(N)$ gauge invariance may be approached by twisting supersymmetry and requiring invariance under the scalar supercharge Q of the resulting twisted gauge theory. In this method, the $16 = 2^4$ supersymmetric charges $(Q_\alpha^i, Q_{\dot{\alpha}i})$, transforming in the spinorial representation of $SO_E(4) \times SO_R(6)$, are thought of in terms of $2^2 \times 2^2$ matrix $\mathbb{Q}_{4 \times 4}$ that can be expanded on products of 4×4 gamma γ^μ matrices; for a general review see [1, 10] and [11]-[14] for related works. The expansion of $\mathbb{Q}_{4 \times 4}$ leads, on one hand, to the integral spin decomposition

$$\mathbb{Q}_{4 \times 4} = IQ + \gamma^\mu Q_\mu + \gamma^{[\mu\nu]} Q_{\mu\nu} + \gamma^\mu \gamma_5 \tilde{Q}_\mu + \gamma_5 \tilde{Q} \quad (1.1)$$

where the 16 supercharges are split as $16 = 1 + 4 + 6 + 4 + 1$; and, on the other hand, to a remarkable packaging of the field spectrum of the twisted 4D $\mathcal{N} = 4$ SYM theory into $SU(5) \times U(1)$ representations like

$$\begin{aligned} \text{bosons} & : 10 \rightarrow 5 \oplus \bar{5} \\ \text{fermions} & : 16 \rightarrow 1 \oplus \bar{5} \oplus 10 \end{aligned} \quad (1.2)$$

Because of the algebraic property $Q^2 = 0$, the scalar supercharge behaves as a topological object [15, 16]; a feature that allows to: (i) put the fields of the twisted $\mathcal{N} = 4$

¹Euclidian QFTs are generally thought of in terms of a Wick rotation of corresponding Lorentzian QFTs. However this analytic continuation is not a soft operation especially for spinors. This issue is not directly addressed in this paper; but results of the Osterwalder-Schrader (OS) method are used. For more details on this issue, including the OS method and other approaches to overcome difficulties induced by analytic continuation, see [30] and refs therein; see also eq(2.22) to fix the ideas.

supersymmetric Yang Mills on a 4D lattice \mathbb{A}_4^* with a hidden $SU(5)$ symmetry [8, 9]; and (ii) write down a $U(N)$ gauge invariant lattice field action \mathcal{S}_{latt} having, in addition to the $SU(5)$ symmetry of \mathbb{A}_4^* , a BRST symmetry generated by Q governing its quantum properties [9].

In this paper, we borrow this idea to study the lattice version of the class of twisted $3D$ supersymmetric Yang-Mills theories with 8 supercharges having an $SU(3) \times U(1)$ symmetry. This twisted $3D$ supersymmetric YM theory follows from the reduction of chiral $\mathcal{N} = (1, 0)$ SYM in $6D$ and living on a particular $3D$ lattice to be built in the present work (see section 6). Our interest into this class of twisted YM theories has been motivated by the two following:

- 1) extend the approach of [1, 3] to the class of lattice supersymmetric YM models based on twisting SYM theories with 8 supercharges. It turns out that the twisted $3D$ lattice gauge theory is very suggestive; it lives on a particular crystal denoted here as

$$\mathcal{L}_{3D}^{su_3 \times u_1}$$

having a hidden $SU(3) \times U(1)$ symmetry; and given by a non trivial fibration $\mathcal{L}_{2D}^{su_3} \times \mathcal{L}_{1D}^{u_1}$ with 2-dimensional base sublattice $\mathcal{L}_{2D}^{su_3} = \mathbb{A}_2^*$ and fiber $\mathcal{L}_{1D}^{u_1}$ isomorphic to \mathbb{Z} , the set of integers. This kind of fibration, encoded by eq(5.8), allows also to get more insight into literature results; especially in the case of twisted maximal supersymmetry living on the lattice \mathbb{A}_4^* with $SU(5)$ symmetry.

To approach the case of twisted SYM with 8 supercharges, and in a subsequent step the class with 16 supercharges as done in section 8, we develop a method of engineering $(k+1)$ -dimensional crystals with $SU(k) \times U(1)$ symmetry; and use results on the breaking mode of the real $SO(2k)$ euclidian symmetries down to the complex

$$SU(k) \times U(1), \quad k \text{ odd integer}$$

to get the packaging of the twisted fields into representations of $SU(k) \times U(1)$; and also to determine their interpretation on lattice $\mathcal{L}_{kD}^{su_k \times u_1}$ in terms of links and plaquettes.

Recall that $4D$ $\mathcal{N} = 4$ supersymmetric Yang Mills with $SO_E(4) \times SO_R(6)$ symmetry is a maximal supersymmetric YM theory that has the same number of conserved supercharges as $\mathcal{N} = (1, 0)$ SYM in euclidian 10-dimensions with isotropy

symmetry

$$SO_E(10), \quad k = 5$$

Similarly, the twisted $3D \mathcal{N} = 4$ YM theory we are interested in here can be obtained in a quite analogous manner; but by dimensional reduction of the chiral $\mathcal{N} = (1, 0)$ SYM in 6-dimensions with euclidian symmetry

$$SO_E(6), \quad k = 3$$

- 2) explore the role of the extra $U(1)$ symmetry that appears in twisted supersymmetric YM theories; in particular in the case of 8 supercharges with $SU(3) \times U(1)$ symmetry; and also in twisted maximal supersymmetry with an $SU(5) \times U(1)$. A way to exhibit this global abelian invariance is through the breaking of $SO_E(2k)$ down to $SU(k) \times U(1)$, which for $k = 5$ and $k = 3$, read respectively as follows

$$\begin{aligned} SO_E(10) &\rightarrow SU(5) \times U(1) \\ SO_E(6) &\rightarrow SU(3) \times U(1) \end{aligned}$$

Under these symmetry breaking modes, real vector $\underline{\mathbf{2k}}$ and spinorial $\underline{\mathbf{2}}^{k-1}$ representations of $SO_E(10)$ and $SO_E(6)$ decompose as sums of representations with respect to the complex symmetries. For $SO_E(10)$, the decomposition of the 10_v and the 16_s are given by

$$\begin{aligned} SO_E(10) &\rightarrow SU(5) \times U(1) \\ 10_v &: 5_{+2q} + \bar{5}_{-2q} \\ 16_s &: 1_{-5q} + \bar{5}_{+3q} + 10_{-q} \end{aligned}$$

with q a unit $U(1)$ charge; and for the case of $SO_E(6)$, their analogue read like

$$\begin{aligned} SO_E(6) &\rightarrow SU(3) \times U(1) \\ 6_v &: 3_{+2q} + \bar{3}_{-2q} \\ 4_s &: 1_{-3q} + 3_{+q} \end{aligned}$$

These breakings show that in twisted supersymmetric YM theories, the twisted fields and the twisted supersymmetric operators, in particular the BRST charge $Q^{(-kq)}$, are in general sections of a $U(1)$ bundle. This property teaches us in turns that on lattice side $SU(3)$ scalars carrying non trivial $U(1)$ charges have also a non trivial interpretation; they are associated with links along the 1- dimensional fiber and, in some sense, constitute a refining of results of [1, 3] since a similar conclusion is also valid for the case of twisted maximal supersymmetry on the lattice $\mathcal{L}_{5D}^{su_5 \times u_1}$.

In what follows, we focus on the study of the lattice version of twisted $3D \mathcal{N} = 4$ SYM; and, to exhibit the role played by $U(1)$ subsymmetry, we distinguish the two cases: the generic $q \neq 0$ and the singular $q = 0$. A similar analysis can be performed for the case of twisted maximal supersymmetric YM in $5D$ as reported in the section conclusion and comments.

The organization is as follows: In sections 2 and 3, we first review some useful tools on $SO(t, s)$ spinors in diverse dimensions $D = t + s$. Then, we study the twisted $3D \mathcal{N} = 4$ supersymmetric YM theory in continuum. In section 4, we build the action in superspace and derive its component field expression. In section 5, we study the twisted $3D \mathcal{N} = 4$ supersymmetric YM on the base sublattice \mathbb{A}_2^* having $SU(3)$ symmetry and corresponding to the singular limit $q = 0$. In section 6, we study twisted $3D \mathcal{N} = 4$ SYM on the $3D$ crystal $\mathcal{L}_{3D}^{su_3 \times u_1}$ with $SU(3) \times U(1)$ symmetry and $q \neq 0$. In section 7, we build the action of the twisted field on $\mathcal{L}_{3D}^{su_3 \times u_1}$. In section 8, we give a conclusion and make two comments; one on the reduction down to twisted $2D \mathcal{N} = 4$ and the second concerns the extension to $5D \mathcal{N} = 4$ on the lattice $\mathcal{L}_{5D}^{su_5 \times u_1}$ containing \mathbb{A}_4^* as a base sublattice. In section 9, we give an appendix where we give explicit computations and technical details on the construction of gauge covariant superfields.

2 Twisted SYM with 8 supercharges

After recalling useful tools on $SO(t, s)$ spinors in diverse dimensions and briefly describing the reduction of chiral $\mathcal{N} = (1, 0)$ SYM in Lorentzian $6D$ down to $3D$, we study twisted $3D \mathcal{N} = 4$ SYM in continuum. More on continuum and the lattice version of this SYM theory will be developed in next sections.

2.1 Generalities on spinors in D-dimensions

Here we collect some results on $SO(t, s)$ spinors living on the flat space $\mathbb{R}^{(t, s)}$ with space time dimension $D = s + t$ and signature $s - t$ where s and t stand respectively for the numbers of space like and time like directions. A particular interest will be given to the Lorentzian ($t = 1$) and euclidian ($t = 0$) signatures that we are interested in this work.

2.1.1 Classification of $so(t, s)$ spinors

Generally speaking, spinors Ψ_A living on $\mathbb{R}^{(t,s)}$ with metric $\eta_{MN} = \text{diag}(-..-, +..+)$ have complex $2^{\lfloor \frac{D}{2} \rfloor}$ components transforming under the space isotropy symmetry $SO(t, s)$ as

$$\Psi_A \rightarrow \Psi'_A = S_A^B \Psi_B \quad (2.1)$$

with matrix transformation given by

$$\begin{aligned} S &= e^{\frac{i}{4} \omega_{MN} \Sigma^{[MN]}} \\ \Sigma^{[MN]} &= \Gamma^M \Gamma^N - \Gamma^N \Gamma^M \end{aligned} \quad (2.2)$$

In these relations, the $2^{\lfloor \frac{D}{2} \rfloor} \times 2^{\lfloor \frac{D}{2} \rfloor}$ matrices Γ_M are the usual gamma matrices generating the Clifford algebra $Cl(t, s)$ defined by

$$\Gamma_M \Gamma_N + \Gamma_N \Gamma_M = 2\eta_{MN} \quad (2.3)$$

The matrices Γ_M are generally realized in terms of particular monomials of tensor products of the usual 2×2 hermitian Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ that we take as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\varepsilon, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2.4)$$

obeying amongst others the property $\sigma_1^T = \sigma_1, \sigma_3^T = \sigma_3$ but $\sigma_2^T = -\sigma_2$. For the example of the euclidian $(0, 6)$ signature, where the metric η_{MN} coincides with the Kronecker symbol δ_{MN} , the Γ_M 's are 8×8 matrices realized as

$$\begin{aligned} \Gamma_1 &= \sigma_1 \otimes \mathbb{I} \otimes \mathbb{I} \\ \Gamma_2 &= \sigma_2 \otimes \mathbb{I} \otimes \mathbb{I} \\ \Gamma_3 &= \sigma_3 \otimes \sigma_1 \otimes \mathbb{I} \\ \Gamma_4 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{I} \\ \Gamma_5 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \\ \Gamma_6 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \\ \Gamma_7 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \end{aligned} \quad (2.5)$$

with the remarkable properties

$$\begin{aligned} (\Gamma_i)^\dagger &= +\Gamma_i \\ (\Gamma_{2k+1})^T &= -\Gamma_{2k+1} \\ (\Gamma_{2k})^T &= +\Gamma_{2k} \end{aligned} \quad (2.6)$$

For the cases of the Lorentzian (1, 5) and (2, 4) signatures, the realization of the corresponding Γ_M 's is obtained from the euclidian representation by using Wick like rotations as follows

signature (1, 5)	signature (2, 4)	
$\Upsilon_0 = i\Gamma_1$ $\Upsilon_m = \Gamma_{m+1}, \quad m > 0$	$\Upsilon_0 = i\Gamma_1 \quad , \quad \Upsilon_1 = i\Gamma_2$ $\Upsilon_m = \Gamma_{m+1}, \quad m > 1$	(2.7)

The complex spinorial field Ψ_A , to which we refer to as $so(t, s)$ Dirac spinors, exhibits several features whose useful ones are summarized below:

(i) *adjoint spinors*

Along with the complex Ψ , one has three cousin spinors namely Ψ^T , Ψ^* and $\Psi^\dagger = \Psi^{*T}$ respectively associated with the Clifford algebras generated by

$$\Gamma_M^T \quad , \quad \Gamma_M^* \quad , \quad \Gamma_M^\dagger \quad (2.8)$$

and which are related to Γ_M by similarity transformations as given below

$$\begin{aligned} \Gamma_M^\dagger &= (-)^t \mathcal{A} \Gamma_M \mathcal{A}^{-1} \\ \Gamma_M^T &= -\eta \mathcal{C} \Gamma_M \mathcal{C}^{-1} \\ \Gamma_M^* &= -\eta (-)^t \mathcal{B} \Gamma_M \mathcal{B}^{-1} \end{aligned} \quad (2.9)$$

with

$$\begin{aligned} \mathcal{A} &= \Gamma_1 \dots \Gamma_t \\ \mathcal{C}^T &= -\varepsilon \mathcal{C} \\ \mathcal{B}^T &= \mathcal{C} \mathcal{A}^{-1} \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} \mathcal{A} &= \eta^t (-)^{\frac{t(t+1)}{2}} \mathcal{C} \mathcal{A} \mathcal{C}^{-1} \\ \mathcal{B}^* \mathcal{B} &= -\varepsilon \eta^t (-)^{\frac{t(t+1)}{2}} \end{aligned} \quad (2.11)$$

and where $\varepsilon = \pm 1$ and $\eta = \pm 1$. We also have

$$(\mathcal{C} \Gamma_M)^T = \varepsilon \eta \mathcal{C} \Gamma_M \quad (2.12)$$

$$(\mathcal{C} \Gamma_1 \dots \Gamma_m)^T = -\varepsilon \eta^m (-)^{\frac{m(m+1)}{2}} (\mathcal{C} \Gamma_1 \dots \Gamma_m)$$

Notice that for odd dimensions, there is one solution for the matrix \mathcal{C} ; but for even dimensions, we distinguish two kinds of possible C matrices as illustrated below on the example of $D = 6$:

$$\begin{aligned}\mathcal{C}_+ &= \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \quad , \quad \text{for } \eta = +1 \quad , \quad \varepsilon = -1 \\ \mathcal{C}_- &= \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \quad , \quad \text{for } \eta = -1 \quad , \quad \varepsilon = +1\end{aligned}\tag{2.13}$$

satisfying $(\mathcal{C}_\pm)^T = \pm \mathcal{C}_\pm$ and $(\mathcal{C}_\pm)^2 = I_{id}$. Notice moreover that for 6D, both of the matrices \mathcal{C}_+ and \mathcal{C}_- have as product $\varepsilon\eta = -1$; and so

$$(\mathcal{C}\Gamma_M)^T = -\mathcal{C}\Gamma_M\tag{2.14}$$

which is an undesirable property that requires the use of $SU(2)$ symplectic spinors $Q_A^i = (Q_A^1, Q_A^2)$ in order to recover the symmetric feature of the anticommutation relation between the supercharges of supersymmetric YM theory in 6D namely

$$Q_A^i Q_B^j + Q_B^j Q_A^i = \varepsilon^{ij} (\mathcal{C}\Gamma^M)_{AB} P_M\tag{2.15}$$

where ε_{ij} is the usual $2 \times N$ antisymmetric matrix obeying $(\varepsilon_{ij})^* \varepsilon_{jl} = -\delta_l^i$.

(ii) *Weyl spinors*

In odd dimensions, the Dirac fermion Ψ satisfying (2.1-2.2) is an irreducible spinor; but in even dimensions, say $D = 2k$, it can be decomposed into two irreducible Weyl spinors Ψ_L and Ψ_R having each 2^{k-1} complex components.

$$2^k = 2^{k-1} \oplus 2^{k-1}\tag{2.16}$$

The two chiral spinors Ψ_L and Ψ_R are related to the Dirac Ψ through the following projections

$$\begin{aligned}\Psi_L &= \frac{1}{2} (I + \Gamma_{D+1}) \Psi \\ \Psi_R &= \frac{1}{2} (I - \Gamma_{D+1}) \Psi\end{aligned}\tag{2.17}$$

with chirality operator $\Gamma_{D+1} = (-i)^{k+t} \Gamma_1 \dots \Gamma_{2k}$ which, by using the realization (2.5), reads as

$$\begin{aligned}\Gamma_{D+1} &= (-i)^{\frac{D}{2}} \Gamma_1 \dots \Gamma_D \\ &= \underbrace{\sigma_3 \otimes \sigma_3 \otimes \dots \otimes \sigma_3}_k\end{aligned}\tag{2.18}$$

This matrix obeys $\Gamma_{D+1} \Gamma_{D+1} = I$ and is independent of the space time signature. Notice that in even dimensions, the anticommutation relations between two Weyl supercharges say the left one

$$Q_L = \frac{1}{2} (I + \Gamma_{D+1}) Q\tag{2.19}$$

read as follows

$$\{Q_L, Q_L\} = \frac{1 - (-)^{\frac{D}{2}}}{4} (I + \Gamma_{D+1}) \mathcal{C} \Gamma^M P_M \quad (2.20)$$

Therefore, non vanishing anticommutators limits the possible dimensions where $\{Q_L, Q_L\} \neq 0$ since the non vanishing condition requires that $\frac{D}{2}$ has to be odd; i.e:

$$D = 4l + 2, \quad l = 1, 2, \dots \quad (2.21)$$

The leading dimensions where this is possible are $D=2$, $D=6$ and $D=10$.

(iii) reality conditions

Complex Dirac spinors might be also subject to reality conditions such as Majorana or Majorana Weyl conditions. These conditions are not usually possible since reality condition depends both on the space time dimension D and the signature $s - t$. The general result on the possibility of putting a reality condition on spinors in diverse Lorentzian and euclidian dimensions is collected in the following table [17]-[20],

dimension D	Lorentzian $\mathbb{R}^{1,D-1}$	Euclidian \mathbb{R}^D	
1	M	M	
2	MW	M^-	
3	M	SM	
4	M^+	SMW	
5	SM	SM	
6	SMW	M^+	(2.22)
7	SM	M	
8	M^-	MW	
9	M	M	
10	MW	M^-	
11	M	SM	

with M standing for Majorana and MW for Majorana Weyl spinors. We also have M^\pm referring to Majorana spinors with charge conjugation C_\pm . Notice that in the case where there is no Majorana spinor, one can have symplectic Majorana spinors or symplectic Majorana Weyl spinors referred in the table respectively by the symbols SM and SMW .

By symplectic Majorana spinor, we mean a set of $2N$ Dirac spinors $\Psi_A^1, \dots, \Psi_A^{2N}$ constrained as follows

$$(\Psi_A^i)^* = \Omega_{ij} \mathcal{B}_A^B \Psi_B^j \quad (2.23)$$

where Ω_{ij} is the usual $2N \times 2N$ antisymmetric symplectic matrix obeying $(\Omega_{ij})^* \Omega_{jl} = -\delta_l^i$ and where $(\mathcal{B}_A^B)^* \mathcal{B}_C^B = -\delta_C^A$. We also have *SMW* whenever \mathcal{B} and Γ_{D+1} commute knowing that

$$(\Gamma_{D+1})^* = (-)^{t+\frac{D}{2}} \mathcal{B} \Gamma_{D+1} \mathcal{B}^{-1} \quad (2.24)$$

Since $(\Gamma_{D+1})^* = \Gamma_{D+1}$ due to (2.18), it follows that \mathcal{B} and Γ_{D+1} commute for $t + \frac{D}{2} = 0 \pmod{2}$. From the classification table (2.22), we learn a set of interesting features in particular:

- there is no Majorana spinor in euclidian $3D, 4D, 5D$; and nor in the Lorentzian $5D, 6D$ and $7D$. Therefore, when studying the euclidian $3D$ $\mathcal{N} = 4$ SYM and euclidian $5D$ $\mathcal{N} = 4$ SYM, one is constrained to use symplectic Majorana spinors.
- the Majorana and Majorana-Weyl conditions are not preserved by analytic continuation from Lorentzian to euclidian signature. This is a well known problem that has been considered from various view points [30] and refs therein; in particular from the approach of Osterwalder-Schrader where the hermiticity in euclidian space is abandoned [31].

2.1.2 Chiral supersymmetric YM in $6D$ and $10D$

Like for the well known case of Lorentzian maximal SYM theories with 16 real supercharges, supersymmetric QFTs with 8 real supercharges can be formulated in diverse dimensions. These are the $1D$ $\mathcal{N} = 8$, $2D$ $\mathcal{N} = (4, 4)$, $3D$ $\mathcal{N} = 4$, $4D$ $\mathcal{N} = 2$ and $5D$ $\mathcal{N} = 2$ theories; they may be obtained by reduction of the chiral

$$6D, \mathcal{N} = (1, 0)$$

supersymmetric YM theory². This Lorentzian $6D$ SYM theory can be then viewed as the mother of supersymmetric theories with 8 supercharges. From this point of view, chiral $\mathcal{N} = (1, 0)$ SYM in $6D$ shares a kind of maternity property with $\mathcal{N} = (1, 0)$ SYM in $10D$ which is the mother theory of supersymmetric QFTs' having 16 supercharges.

²Notice that for those dimensions D where there is no Majorana spinor like in $D = 1 + 4$ or euclidian $3D$ and $5D$, we shall also use the standard conventional notations $5D$ $\mathcal{N} = 4$ ($3D$ $\mathcal{N} = 4$) to refer to the 16 real (8 real) supercharges although strictly speaking this convention is not rigorous.

Twisted 3D $\mathcal{N} = 4$ SYM

In twisted 3D $\mathcal{N} = 4$ supersymmetric YM theory, the 8 real supersymmetric charges are represented by a complex 2×2 matrix $\mathbb{Q}_{2 \times 2}$ that can be expanded in terms of the 3 Pauli matrices σ^μ as follows

$$\mathbb{Q}_{2 \times 2} = QI + Q_\mu \sigma^\mu \quad (2.25)$$

Similarly as for the case of eq(1.2) of twisted maximal supersymmetric YM theory, the field content of the spectrum of the twisted 3D $\mathcal{N} = 4$ SYM theory can be packaged into $SU(3) \times U(1)$ representations like

$$\begin{aligned} \text{bosons} & : 6 \rightarrow 3 \oplus \bar{3} \\ \text{fermions} & : 4 \rightarrow 1 \oplus 3 \end{aligned} \quad (2.26)$$

Correspondence 3D $\mathcal{N} = 4$ and 5D $\mathcal{N} = 4$

Pushing forward the similarity between twisted SYM with 8 supercharges and twisted maximal supersymmetric YM (*see footnote 2*), one finds the following correspondence to be established throughout this study:

- *twisted SYM with 16 supercharges*

	lattice	hidden symmetry
$5D \mathcal{N} = 4$	$\mathcal{L}_{5D}^{su_5 \times u_1}$	$SU(5) \times U(1)$
$4D \mathcal{N} = 4$	$\mathcal{L}_{4D}^{su_5} = \mathbb{A}_4^*$	$SU(5)$

(2.27)

with 5- dimensional lattice $\mathcal{L}_{5D}^{su_5 \times u_1}$ given by the fibration

$$\begin{array}{ccc} \mathcal{L}_{1D}^{u_1} & \rightarrow & \mathcal{L}_{5D}^{su_5 \times u_1} \\ & & \downarrow \\ & & \mathcal{L}_{4D}^{su_5} \end{array} \quad (2.28)$$

The base sublattice $\mathcal{L}_{4D}^{su_5}$ is given by the 4- dimensional lattice [1, 3]

$$\mathcal{L}_{4D}^{su_5} = \mathbb{A}_4^* \quad (2.29)$$

generated by the 4 fundamental weight vectors

$$\vec{\Omega}_1, \quad \vec{\Omega}_2, \quad \vec{\Omega}_3, \quad \vec{\Omega}_4 \quad (2.30)$$

of the $SU(5)$ symmetry group. These weight vectors are the dual of the 4 simple roots of the Lie algebra of $SU(5)$;

$$\vec{a}_1, \quad \vec{a}_2, \quad \vec{a}_3, \quad \vec{a}_4 \quad (2.31)$$

generating the 4-dimensional root lattice \mathbb{A}_4 of $SU(5)$. So the crystal \mathbb{A}_4^* is the dual of \mathbb{A}_4 ; see later on for further details and [21]-[29] for related constructions.

- *twisted SYM with 8 supercharges*

	lattice	hidden symmetry
$3D \mathcal{N} = 4$	$\mathcal{L}_{3D}^{su_3 \times u_1}$	$SU(3) \times U(1)$
$2D \mathcal{N} = 4$	$\mathcal{L}_{2D}^{su_3}$	$SU(3)$

(2.32)

with

$$\begin{array}{ccc} \mathcal{L}_{1D}^{u_1} & \rightarrow & \mathcal{L}_{3D}^{su_3 \times u_1} \\ & & \downarrow \\ & & \mathcal{L}_{2D}^{su_3} \end{array} \quad (2.33)$$

and base sublattice given by the 2- dimensional lattice [33]-[34]

$$\mathcal{L}_{2D} = \mathbb{A}_2^* \quad (2.34)$$

generated by the 2 fundamental weight vectors

$$\vec{\omega}_1 \quad , \quad \vec{\omega}_2 \quad (2.35)$$

of the $SU(3)$ symmetry. These weight vectors are the dual of the 2 simple roots $\vec{\alpha}_1, \vec{\alpha}_2$ of $SU(3)$; then \mathbb{A}_2^* is the dual of the 2-dimensional root lattice \mathbb{A}_2 of $SU(3)$ which may be thought of as the 2D honeycomb, see fig. 1 for illustration.

In what follows, we study the twisted $3D \mathcal{N} = 4$ SYM in continuum. First, we describe some special features on SYM in $6D$; then we derive the $SU(3) \times U(1)$ covariant spectrum

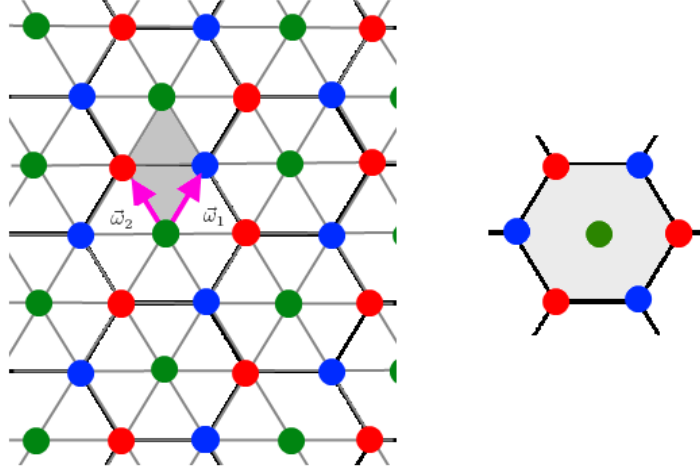


Figure 1: the 2D lattice \mathbb{A}_2^* generated by $\vec{\omega}_1, \vec{\omega}_2$; the 2 fundamental weight vectors of $SU(3)$. Each (green) node in \mathbb{A}_2^* has 3 + 3 first nearest neighbors forming respectively a triplet (red sites) and an anti-triplet (blue sites) of $SU(3)$.

of twisted $\mathcal{N} = 4$ SYM in $3D$. Next, we give the twisted $\mathcal{N} = 4$ superalgebra having an $SU(3) \times U(1)$ isotropy symmetry with supersymmetric generators as

$$Q^{(+3q)} \quad , \quad Q_a^{(-q)} \quad (2.36)$$

transforming respectively as a complex scalar and a complex triplet of $SU(3)$. After, we use superspace method to realize the scalar supersymmetric charge $Q^{(+3q)}$ which may be also thought of as a BRST charge operator.

2.2 Twisted $\mathcal{N} = 4$ SYM in $3D$ from $\mathcal{N} = 1$ SYM₆

In Lorentzian $6D$ one distinguishes 3 kinds of supersymmetric YM theories: two of them have 16 real conserved supercharges and the third one has 8 real supercharges [35]-[37]. The field theories having 16 supercharges are given by the well known non chiral $6D$ $\mathcal{N} = (1, 1)$ and the chiral $6D$ $\mathcal{N} = (2, 0)$. The field theory with 8 supercharges is given by the chiral $6D$ $\mathcal{N} = (1, 0)$ SYM or equivalently $\mathcal{N} = (0, 1)$; it is the gauge theory we consider below.

2.2.1 the $6D$ $\mathcal{N} = 1$ vector multiplet

First, recall that in Lorentzian $6D$ there are only Weyl Ψ_{6D}^W and Dirac fermions $\Psi_{6D}^{Dirac} = (\Psi_{6D}^{WL}, \Psi_{6D}^{WR})$; so that the smallest supermultiplet contains a left Ψ_{6D}^{WL} fermion, transform-

ing in the $SO(1, 5)$ spinor representation 4_+ , or right Ψ_{6D}^{WR} Weyl fermion transforming in 4_- . Recall also that in the language of $4D$ fermions, a Weyl spinor in $6D$, say the left one,

$$\Psi_{6D}^{WL} \sim 4_+$$

having 4 complex (8 real) degrees of freedom, is made of a dotted and an undotted $4D$ Weyl spinors as follows

$$\Psi_{6D}^{WL} = \left(\xi^{\alpha+}, \bar{\lambda}_{\dot{\alpha}}^- \right) \quad , \quad \alpha = 1, 2 \quad (2.37)$$

Its complex adjoint is

$$(\Psi_{6D}^{WL})^c = \left(\lambda^{\alpha+}, \bar{\xi}_{\dot{\alpha}}^- \right) \sim 4_+^c \quad (2.38)$$

and has the same 6D chirality as 4_+ . The \pm charges carried by the fields refer to quantum numbers of $SO_R(2) \sim U_R(1)$ resulting from the reduction of the $SO(1, 5)$ Lorentz symmetry down to $SO(1, 3) \times SO_R(2)$. The corresponding 6D Weyl right fermion

$$\Psi_{6D}^{WR} \sim 4_-$$

with negative $6D$ chirality is given by

$$\begin{aligned} \Psi_{6D}^{WR} &= \left(\xi^{\alpha-}, \bar{\lambda}_{\dot{\alpha}}^+ \right) \\ (\Psi_{6D}^{WR})^c &= \left(\lambda^{\alpha-}, \bar{\xi}_{\dot{\alpha}}^+ \right) \end{aligned} \quad (2.39)$$

The chiral $6D$ $\mathcal{N} = (1, 0)$ supersymmetric YM theory has two types of supermultiplets describing matter and gauge fields with on shell degrees of freedom as follows:

(a) *6D hypermultiplets*

These supermultiplets describe matter; they have 2 complex (4 real) scalars and a 6D Weyl spinor

$$\mathcal{H}_{6D} : \left(\frac{1}{2}, 0^4 \right)_{6D} \quad (2.40)$$

they may belong to any representation of the gauge symmetry including complex ones; see [38]-[17] for other properties.

(b) *6D vector multiplets* \mathcal{V}_{6D}

These multiplets have a gauge field and a Weyl spinor

$$\mathcal{V}_{6D}^{\mathcal{N}=(1,0)} : \left(1, \frac{1}{2} \right)_{6D} \quad (2.41)$$

they transform in the adjoint representation of the gauge symmetry. Below, we refer to these supermultiplets like

$$\mathcal{V}_{6D}^{\mathcal{N}=1} = (\mathcal{A}_M, \psi^A)_{6D} \quad (2.42)$$

where the field \mathcal{A}_M is the 6D hermitian gauge field and ψ^A the complex 4- dimension Weyl spinor of

$$SO(1, 5) \simeq SU^*(4) \quad (2.43)$$

the fields \mathcal{A}_M and ψ^A are valued in the Lie algebra of the $U(N)$ gauge symmetry.

2.2.2 Reduction from 6D to 3D and twisting

We give two approaches to build the twisted field spectrum of $3D \mathcal{N} = 4$ supersymmetric YM that follows from the reduction of (2.41-2.42). We also comment on the link between the two methods.

(1) first approach

This approach is a rephrasing of eq(2.25); it involves two steps: (i) dimension reduction from 6D to 3D; (ii) twisting the symmetries resulting from the breaking of $SO(1, 5)$.

Under the reduction of the chiral 6D $\mathcal{N} = (1, 0)$ supersymmetric YM theory down to the 3D space, the $SO(1, 5)$ breaks down to

$$SO(1, 2) \times SO_R(3)$$

and so the 6 local coordinates X^M of $\mathbb{R}^{1,5}$ decompose like

$$(x^\mu, y^m)$$

with $x^\mu \in \mathbb{R}^{1,2}$ and $y^m \in \mathbb{R}_R^3$ with respective isotropy symmetries $SO(1, 2)$ and $SO_R(3)$. Similarly, the on shell $4 + 4$ real degrees of freedom of the 6D chiral $\mathcal{N} = (1, 0)$ gauge multiplet (2.41), decomposes into a gauge field A_μ , three real scalars ϕ_m and 4 Majorana spinors $\psi^{\alpha 1}, \dots, \psi^{\alpha 4}$. In the euclidian version of this theory, the $SO(1, 2) \times SO_R(3)$ isotropy gets mapped to the compact $SO_E(3) \times SO_R(3)$ and the 4 Majorana spinors $\psi^{\alpha I}$ into 2 complex Dirac spinors $\xi^{\alpha 1}, \xi^{\alpha 2}$ like

$$\left(1, \frac{1^2}{2_{Dirac}}, 0^3\right)_{3D} = (A_\mu, \xi^{\alpha i}, \phi_m)_{3D} \quad (2.44)$$

with:

- the field A_μ being a real 3D gauge field transforming as $(3, 1)$ under $SO_E(3) \times SO_R(3)$,

- the fields $\xi^{\alpha i}$ are complex fermions of transforming $(2, 2)$ spinors of $SO_E(3) \times SO_R(3) \simeq SU_E(2) \times SU_R(2)$,
- the fields ϕ_m are 3 real scalars transforming as $(1, 3)$ under $SO_E(3) \times SO_R(3)$.

Notice that in practice these fields should be taken as functions depending only on the x coordinates;

$$A_\mu^{(0)} = A_\mu(x), \quad \xi_{(0)}^{\alpha i} = \xi^{\alpha i}(x), \quad \dots \quad (2.45)$$

but generally speaking they are functions of both coordinates (x, y) ;

$$A_\mu = A_\mu(x, y), \quad \xi^{\alpha i} = \xi^{\alpha i}(x, y), \quad \dots \quad (2.46)$$

By taking y^m as the coordinates of a real 3-torus \mathbb{T}^3 with large volume

$$\frac{1}{(2\pi l)^3} \int_{\mathbb{T}^3} d^3 y = 1, \quad \text{vol}(\mathbb{T}^3) = (2\pi l)^3 \quad (2.47)$$

one may expand these fields into infinite harmonic series like

$$F(x, y) = \sum_{n_1, n_2, n_3} e^{in_m \frac{y^m}{2\pi l}} F^{(n_1, n_2, n_3)}(x) \quad (2.48)$$

where eqs(2.45) appear as the zero mode of the expansions and the extra others as massive modes that break gauge symmetry in the restricted real 3D.

Under twisting, the quantum numbers of $SO_E(3)$ and $SO_R(3)$ groups are identified and the $SO_E(3) \times SO_R(3)$ symmetry is reduced down to the diagonal

$$SO(3) = \frac{SO_E(3) \times SO_R(3)}{SO'(3)} \quad (2.49)$$

As a consequence of the twisting, the fields of the chiral $6D$ $\mathcal{N} = (1, 0)$ gauge multiplet (2.41) are mapped to the twisted ones

fields	:	twisted fields	SO(3) repres	
A_μ		A_μ	3	
ϕ_m		B_μ	3	
$\xi^{\alpha \pm}$		ξ, ξ^μ	$1 \oplus 3$	(2.50)

where now we have:

- two gauge fields A_μ, B_μ that we combine into a complex gauge field and its adjoint like

$$\begin{aligned}\mathcal{G}^\mu &= A_\mu + iB_\mu \\ \bar{\mathcal{G}}_\mu &= A_\mu - iB_\mu\end{aligned}\tag{2.51}$$

- four complex fermionic fields ξ, ξ^μ transforming respectively as a singlet and triplet of $SO(3)$.

(2) *second approach*

This approach involves one step; and, in some sense, is a direct method. The idea of this way of doing relies on the fact that since the fields \mathcal{G}^μ and ξ, ξ^μ are complex fields, one may be tempted to use complex groups to deal with them; this extension can be implemented by considering other breaking modes of the $SO_E(6)$ isotropy symmetry of the euclidian space time \mathbb{R}^6 (following from the Wick rotation of $\mathbb{R}^{1,5}$); in particular

$$\begin{aligned}SO_E(6) &\longrightarrow SU(3) \times U(1) \\ 6_v &\sim 3_{+2q} + \bar{3}_{-2q} \\ 4_s &\sim 3_{+q} + 1_{-3q}\end{aligned}\tag{2.52}$$

where q is a unit charge of the abelian $U(1)$ factor that can be fixed to a number q_0 . But here we will keep it free for later use when considering the singular limit $q = 0$.

Under the breaking mode (2.52), the euclidian space \mathbb{R}^6 , parameterized by the local coordinates $X^M = (x^\mu, y^\mu)$ with $y^\mu = x^{\mu+3}$, get mapped to the complex \mathbb{C}^3 with local coordinates

$$z^a = x^a + iy^a\tag{2.53}$$

where (x^a) the coordinates of the real space \mathbb{R}^3 and (y^a) the coordinates of the internal space \mathbb{R}_{int}^3 . Moreover, the fields of the multiplet (2.41) decompose as follows

$$\begin{aligned}SO_E(6) &\longrightarrow SU(3) \times U(1) \\ \mathcal{A}_M &: \mathcal{G}^{a(-2q)}, \quad \bar{\mathcal{G}}_a^{(+2q)} \\ \Psi^A &: \psi^{a(+q)}, \quad \psi^{(-3q)}\end{aligned}\tag{2.54}$$

and may be treated in general as functions of (z, \bar{z}) ; this property will be manifested on the lattice side through orientations of the links; complex p-tensors and their duals are associated with p-plaquettes with opposite orientations.

Moreover, comparing eqs(2.50-2.51) with eq(2.52-2.54), we end with the following results:

(a) the spectrum of twisted fields of the two approaches are quasi the same; the main difference is that (2.52) depend on the extra charge q and transform in $SU(3)$ representations rather than $SO(3)$.

(b) Eqs(2.50-2.51) are recovered from eqs(2.52-2.54) by taking the limit

$$q \rightarrow 0 \tag{2.55}$$

and restricting the complex $SU(3) \times U(1)$ symmetry down to the real $SO(3)$ which may be thought of as its "real part". In practice, this corresponds to dropping out the y -dependence into the fields and using (2.47) to integrate it out in the field action.

3 Twisted $3D \mathcal{N} = 4$ algebra and superfields

We first give the basic anticommutators defining this superalgebra; then we describe the general structure of twisted superspace and superfields.

3.1 Twisted $\mathcal{N} = 4$ supersymmetry in $3D$

For a generic charge q , the twisted $3D \mathcal{N} = 4$ supersymmetric algebra is generated by

$$Q^{(+3q)} \quad , \quad Q_a^{(-q)} \quad , \quad P_a^{(+2q)} \tag{3.1}$$

and

$$P^{a(-2q)} \tag{3.2}$$

having no supersymmetric partner; a property that makes *asymmetric* the formulation of twisted supersymmetric YM.

3.1.1 *anticommutators*

These operators transform under $U(1) \times SU(3)$ as in (2.52); and satisfy the following basic anticommutation relations,

$$\begin{aligned} \left\{ Q^{(+3q)}, Q_a^{(-q)} \right\} &= 2P_a^{(+2q)} \\ \left\{ Q_a^{(-q)}, Q_b^{(-q)} \right\} &= 0 \end{aligned} \quad (3.3)$$

together with the topological one

$$\{ Q^{(+3q)}, Q^{(+3q)} \} = 0 \quad (3.4)$$

and

$$\begin{aligned} [Q^{(+3q)}, P_a^{(+2q)}] &= [Q_a^{(-q)}, P_b^{(+2q)}] = 0 \\ [Q^{(+3q)}, P^{a(-2q)}] &= [Q_a^{(-q)}, P^{b(-2q)}] = 0 \end{aligned} \quad (3.5)$$

These graded commutation relations preserve the $U(1)$ charge and are covariant under $SU(3)$ symmetry.

3.1.2 *twisted superspace and superderivatives*

The twisted $3D \mathcal{N} = 4$ superalgebra may be realized in superspace by using complex bosonic and fermionic coordinates

$$z^{a(-2q)}, \quad z_a^{(+2q)}, \quad \theta^{(-3q)}, \quad \vartheta^{a(+q)} \quad (3.6)$$

with

$$(z_a^{(+2q)})^\dagger = z^{a(-2q)} \quad (3.7)$$

Using the usual supersymmetric covariant derivatives $D^{(+3q)}$ and $D_a^{(-q)}$, instead of the supercharges $Q^{(+3q)}$ and $Q_a^{(-q)}$, a suitable superspace representation of the twisted superalgebra (3.3) is given by

$$\begin{aligned} D^{(+3q)} &= \frac{\partial}{\partial \theta^{(-3q)}} \\ D_a^{(-q)} &= \frac{\partial}{\partial \vartheta^{a(+q)}} + 2\theta^{(-3q)} \partial_a^{(+2q)} \end{aligned} \quad (3.8)$$

with

$$\begin{aligned} P^{a(-2q)} &= \partial^{a(-2q)} = \frac{\partial}{\partial z_a^{(+2q)}} \\ \bar{P}_a^{(+2q)} &= \partial_a^{(+2q)} = \frac{\partial}{\partial z^{a(-2q)}} \end{aligned} \quad (3.9)$$

To implement gauge interactions, these superspace derivatives are covariantized by introducing gauge connexions as follows

$$\begin{aligned}
\mathcal{D}^{(+3q)} &= D^{(+3q)} + ig_{YM} \Upsilon^{(+3q)} \\
\mathcal{D}_a^{(-q)} &= D_a^{(-q)} + ig_{YM} \Upsilon_a^{(-q)} \\
L_a^{(+2q)} &= \partial_a^{(+2q)} + ig_{YM} V_a^{(+2q)} \\
L^{a(-2q)} &= \partial^{a(-2q)} + ig_{YM} U^{a(-2q)}
\end{aligned} \tag{3.10}$$

These extended superderivatives are needed for building the gauge covariant superfields $\Phi_i^{(q_i)}$ of the twisted YM theory to be considered later.

3.2 Superfields of twisted 3D $\mathcal{N} = 4$ SYM

To build the field action $\mathcal{S}_{twisted}$ of the twisted 3D $\mathcal{N} = 4$ supersymmetric YM theory, we require invariance under the three following symmetries:

- (a) the gauge symmetry which we take as $U(N)$,
- (b) the scalar supersymmetric charge $Q^{(+3q)}$ or equivalently $D^{(+3q)}$; and,
- (c) the $U(1) \times SU(3)$ space isotropy symmetry

First, observe that the gauge invariant action

$$\mathcal{S}_{twisted} = \frac{1}{l^3} \int \mathbf{L}_{twist} \tag{3.11}$$

with the scalar supercharge $Q^{(+3q)}$ manifestly exhibited reads in superspace as follows

$$\mathbf{L}_{twist} = \left(\int d\theta^{(-3q)} \mathcal{L}^{(-3q)} \right)_{\vartheta^{a(+q)}=0} \tag{3.12}$$

The scale factor $\frac{1}{l^3}$ is as in eq(2.47). The superspace density $\mathcal{L}^{(-3q)}$ transforms in the $U(1) \times SU(3)$ representation 1_{-3} ; that is having -3 unit charges under $U(1)$, and has the form

$$\mathcal{L}^{(-3q)} = Tr \left(\mathbf{L}_{twist}^{(-3q)} \right) \tag{3.13}$$

with the $N \times N$ superfield matrix

$$\mathbf{L}_{twist}^{(-3q)} = \mathbf{L}_{twist}^{(-3q)}(\Phi) \tag{3.14}$$

The $\mathbf{L}_{twist}^{(-3q)}$ depends on a set of superfields

$$\Phi_i^{(q_i)} = \Phi_i^{(q_i)}(z, \theta, \vartheta) \tag{3.15}$$

that describe the off shell degrees of freedom of twisted $3D \mathcal{N} = 4$ supersymmetric YM theory. Below, we describe this set of superfields; for more details on the explicit derivation see the analysis given in the appendix.

3.2.1 Gauge covariance

The $U(N)$ gauge symmetry of the action (3.12) acts on the superfield matrix density $\mathbf{L}_{twist}^{(-3)}$ like

$$\mathbf{L}_{twist}^{(-3q)} \rightarrow \mathbf{g} \mathbf{L}_{twist}^{(-3q)} \mathbf{g}^{-1} \quad (3.16)$$

since

$$Tr \left(\mathbf{L}_{twist}^{(-3q)} \right) = Tr \left(\mathbf{g} \mathbf{L}_{twist}^{(-3q)} \mathbf{g}^{-1} \right) \quad (3.17)$$

where, for convenience as described in the appendix, the matrix element \mathbf{g} is chosen as follows

$$\mathbf{g} = \mathbf{g}(z, \bar{z}, \vartheta^{a(+q)}) \quad , \quad D^{(+3q)} \mathbf{g} = 0 \quad (3.18)$$

Notice that \mathbf{g} depend on $z, \bar{z}, \vartheta^{a(+q)}$ but has no

$$\theta^{(-3q)}$$

The property (3.17) is ensured by requiring the superfields $\Phi_i^{(q_i)}$ to be also gauge covariant; this means that under a generic gauge symmetry transformation \mathbf{g} , we have

$$\Phi_i^{(q_i)} \rightarrow \mathbf{g} \Phi_i^{(q_i)} \mathbf{g}^{-1} \quad (3.19)$$

General results on covariant formulation of supersymmetric YM theories in superspace [43] applied to our present study lead to the following set of gauge covariant superfields.

Fermionic sector	:	$\Psi^{(-3q)}$	$\Psi^{a(+q)}$	$\Phi_{ab}^{(+q)}$
$SU(3) \times U(1)$:	1_{-3q}	3_{+q}	3_{+q}
scale mass dim		1	1	1

(3.20)

Bosonic sector	:	$\mathbb{J}^{(0)}$, $\mathbb{E}^{ab(-4q)}$	$\mathbb{F}_{ab}^{(+4q)}$
$SU(3) \times U(1)$:	1_0	$\bar{3}_{-4q}$	3_{+4q}
scale mass dim		$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$

built out of commutators of the gauge covariant superderivatives

$$\begin{aligned}
\Psi^{(-3q)} &= \frac{1}{ig_{YM}} \left[\mathcal{D}_a^{(-q)}, L^{a(-2q)} \right] \\
\Psi^{a(+q)} &= \frac{1}{ig_{YM}} \left[\mathcal{D}^{(+3q)}, L^{a(-2q)} \right] \\
\Phi_{ab}^{(+q)} &= \frac{1}{ig_{YM}} \left[\mathcal{D}_a^{(-q)}, L_b^{(+2q)} \right]
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
\mathbb{J}^{(0)} &= \frac{1}{ig_{YM}} \left[L_a^{(+2q)}, L^{a(-2q)} \right] \\
\mathbb{E}^{ab(-4q)} &= \frac{1}{ig_{YM}} \left[L^{a(-2q)}, L^{b(-2q)} \right] \\
\mathbb{F}_{ab}^{(+4q)} &= \frac{1}{ig_{YM}} \left[L_a^{(+2q)}, L_b^{(+2q)} \right]
\end{aligned} \tag{3.22}$$

with gauge coupling constant g_{YM} scaling like $(mass)^{\frac{1}{2}}$. Notice that as far as superfields with scaling dimension as $(mass)^1$ are concerned, eqs(3.20) may also contain the fermionic superfield

$$\Psi_a^{(+5q)} = \frac{1}{ig_{YM}} \left[\mathcal{D}^{(+3q)}, L_a^{(+2q)} \right] \tag{3.23}$$

it is constrained to be equal to zero in our construction. The above fermionic and bosonic gauge covariant superfields obey as well constraint relations; in particular

$$\begin{aligned}
\mathcal{D}^{(+3q)} \Psi^{(-3q)} &= 2\mathbb{J}^{(0)} - \mathcal{D}_a^{(-q)} \Psi^{a(+q)} \\
\mathcal{D}^{(+3q)} \mathbb{E}^{ab(-4q)} &= L^{a(-2q)} \Psi^{b(+q)} - L^{b(-2q)} \Psi^{a(+q)} \\
\mathcal{D}^{(+3q)} \Phi_{ab}^{(+q)} &= \mathbb{F}_{ab}^{(+4q)}
\end{aligned} \tag{3.24}$$

and remarkably

$$\begin{aligned}
\mathcal{D}^{(+3q)} \Psi^{a(+q)} &= 0 \\
\mathcal{D}^{(+3q)} \mathbb{F}_{ab}^{(+4q)} &= 0 \\
L_b^{(+2q)} \Psi^{(-3q)} &= L^{a(-2q)} \Phi_{ab}^{(+q)}
\end{aligned} \tag{3.25}$$

To deal with these constraint relations, it is helpful to use the following θ - expansions

$$\begin{aligned}
\Psi^{a(+q)} &= \psi^{a(+q)} + \theta^{(-3q)} f^{a(+4q)} \\
\Psi^{(-3q)} &= \psi^{(-3q)} + \theta^{(-3q)} F^{(0)} \\
\Phi_{ab}^{(+q)} &= \phi_{ab}^{(+q)} + \theta^{(-3q)} \mathcal{F}_{ab}^{(+4q)} \\
\mathbb{J}^{(0)} &= \mathcal{F}^{(0)} + \theta^{(-3q)} \nabla_a^{(+2q)} \psi^{a(+q)} \\
\mathbb{E}^{ab(-4q)} &= \mathcal{F}^{ab(-4q)} + \theta^{(-3q)} \left[\nabla^{a(-2q)} \psi^{b(+q)} - \nabla^{b(-2q)} \psi^{a(+q)} \right] \\
\mathbb{F}_{ab}^{(+4q)} &= \mathcal{F}_{ab}^{(+4q)} + \theta^{(-3q)} \mathcal{Z}_{ab}^{(+7q)}
\end{aligned} \tag{3.26}$$

with the fields

$$\psi^{(-3q)}, \quad \psi^{a(+q)} \tag{3.27}$$

being the twisted fermionic fields of eqs(2.54); the bosonic fields $f^{a(+4q)}$ and $F^{(0)}$ scaling as $(mas)^{\frac{3}{2}}$ are auxiliary fields; and finally

$$\mathcal{F}^{(0)}, \quad \mathcal{F}^{ab(-4q)}, \quad \mathcal{F}_{ab}^{(+4q)} \tag{3.28}$$

are as follows

$$\begin{aligned}
\mathcal{F}_{ab}^{(+4q)} &= \frac{1}{ig_{YM}} \left[\nabla_a^{(+2q)}, \nabla_b^{(+2q)} \right] \\
\mathcal{F}^{ab(-4q)} &= \frac{1}{ig_{YM}} \left[\nabla^{a(-2q)}, \nabla^{b(-2q)} \right] \\
\mathcal{F}^{(0)} &= \frac{1}{ig_{YM}} \left[\nabla_a^{(+2q)}, \nabla^{a(-2q)} \right]
\end{aligned} \tag{3.29}$$

with the gauge covariant $\nabla^{a(-2q)}, \nabla_a^{(+2q)}$ derivatives given by

$$\begin{aligned}
\nabla^{a(-2q)} &= \partial^{a(-2q)} + ig_{YM} \mathcal{G}^{a(-2q)} \\
\nabla_a^{(+2q)} &= \partial_a^{(+2q)} + ig_{YM} \mathcal{G}_a^{(+2q)}
\end{aligned} \tag{3.30}$$

with $\mathcal{G}_a^{(+2q)}, \mathcal{G}^{a(-2q)}$ as in (2.54).

We also have the relations

$$\begin{aligned}
\nabla_a^{(+2q)} \psi^{a(+q)} &= \partial_a^{(+2q)} \psi^{a(+q)} + ig_{YM} \left[\mathcal{G}_a^{(+2q)}, \psi^{a(+q)} \right] \\
\nabla^{a(-2q)} \psi^{b(+q)} &= \partial^{a(-2q)} \psi^{b(+q)} + ig_{YM} \left[\mathcal{G}^{a(-2q)}, \psi^{b(+q)} \right]
\end{aligned} \tag{3.31}$$

Notice that the constraint relation $\mathcal{D}^{(+3q)}\Psi^{a(+q)} = 0$, which by using the gauge fixing described in the appendix reads also like

$$D^{(+3q)}\Psi^{a(+q)} = 0 \quad (3.32)$$

is solved as follows

$$\Psi^{a(+q)} = \psi^{a(+q)} \quad , \quad f^{a(+4q)} = 0 \quad (3.33)$$

This solution shows that $\psi^{a(+q)}$ is a supersymmetric invariant field in agreement with the θ - expansion of the the gauge superfield $V^{a(-2q)}$ involved in (3.10),

$$U^{a(-2q)} = \mathcal{G}^{a(-2q)} + \theta^{(-3q)}\psi^{a(+q)} \quad (3.34)$$

Similarly, we have for the constraint $D^{(+3q)}\mathbb{F}_{ab}^{(+q)} = 0$, the following

$$\mathbb{F}_{ab}^{(+4q)} = \mathcal{F}_{ab}^{(+4q)} \quad , \quad \varkappa_{ab}^{(+7q)} = 0 \quad (3.35)$$

showing that $\mathcal{F}_{ab}^{(+q)}$ is a supersymmetric invariant field in agreement with the θ - expansion of the the gauge superfield $\Upsilon_a^{(-q)}$ involved in eqs(3.10) namely

$$\Upsilon_a^{(-q)} = \gamma_a^{(-q)} + \theta^{(-3q)}\mathcal{G}_a^{(+2q)} \quad (3.36)$$

From this relation, we also learn that $\mathcal{G}_a^{(+2q)}$ is supersymmetric invariant and so the superfield $V_a^{(+2q)}$ appearing in (3.10) has no $\theta^{(-3q)}$ dependence and then should be as

$$V_a^{(+2q)} = \mathcal{G}_a^{(+2q)} \quad (3.37)$$

Regarding the constraint relation $L_b^{(+2q)}\Psi^{(-3q)} = L^{a(-2q)}\Phi_{ab}^{(+q)}$, we use the θ - expansions of the superfields to end, on one hand, with

$$\begin{aligned} L_b^{(+2q)}\Psi^{(-3q)} &= \partial_b^{(+2q)}\Psi^{(-3q)} + ig_{YM} \left[\mathcal{G}_a^{(+2q)}, \Psi^{(-3q)} \right] \\ &= \nabla_a^{(+2q)}\psi^{(-3q)} + ig_{YM}\theta^{(-3q)}\nabla_a^{(+2q)}F^{(0)} \end{aligned} \quad (3.38)$$

and, on the other hand, with

$$\begin{aligned} L^{a(-2q)}\Phi_{ab}^{(+q)} &= \partial^{a(-2q)}\Phi_{ab}^{(+q)} + ig_{YM} \left[U^{a(-2q)}, \Phi_{ab}^{(+q)} \right] \\ &= \nabla^{a(-2q)}\phi_{ab}^{(+q)} + ig_{YM}\theta^{(-3q)} \left(\left[\mathcal{G}^{a(-2q)}, \mathcal{F}_{ab}^{(+4q)} \right] + \left\{ \psi^{a(+q)}, \phi_{ab}^{(+q)} \right\} \right) \end{aligned}$$

By equating, we obtain

$$\begin{aligned}\nabla_b^{(+2q)}\psi^{(-3q)} &= \nabla^{a(-2q)}\phi_{ab}^{(+q)} \\ \left[\mathcal{G}_b^{(+2q)}, F^{(0)}\right] &= \left[\mathcal{G}^{a(-2q)}, \mathcal{F}_{ab}^{(+4q)}\right] + \left\{\psi^{a(+q)}, \phi_{ab}^{(+q)}\right\}\end{aligned}\tag{3.39}$$

Under gauge transformations with matrix element \mathbf{g} chosen, for simplicity, as

$$\mathbf{g} = \mathbf{g}\left(z, \bar{z}, v^{a(+q)}\right) \quad , \quad D^{(+3q)}\mathbf{g} = 0\tag{3.40}$$

the superfields (3.20) satisfy

$$\begin{aligned}\Psi^{(-3q)} &\rightarrow \mathbf{g}\Psi^{(-3q)}\mathbf{g}^{-1} \\ \Phi_{ab}^{(+q)} &\rightarrow \mathbf{g}\Phi_{ab}^{(+q)}\mathbf{g}^{-1}\end{aligned}\tag{3.41}$$

and

$$\begin{aligned}\mathbb{E}^{ab(-4q)} &\rightarrow \mathbf{g}\mathbb{E}^{ab(-4q)}\mathbf{g}^{-1} \\ \mathbb{J}^{(0)} &\rightarrow \mathbf{g}\mathbb{J}^{(0)}\mathbf{g}^{-1}\end{aligned}\tag{3.42}$$

3.2.2 *Supersymmetric transformations*

First, we give the supersymmetric transformations of the on shell degrees of freedom of eq(2.54); then we consider the transformations of a particular set of off shell ones.

- *On shell multiplet*

Using the equations of motion of the on shell twisted fields; in particular $\nabla_a^{(+2q)}\psi^{a(+q)} = 0$ and $\nabla_a^{(+2q)}\psi^{(-3q)} = 0$, we can write down the supersymmetric transformations generated by the scalar operator $Q^{(+3q)}$; they are given by

$$\begin{aligned}Q^{(+3q)}\mathcal{G}^{a(-2q)} &= \psi^{a(+q)} \\ Q^{(+3q)}\psi^{a(+q)} &= 0 \\ Q^{(+3q)}\mathcal{G}_a^{(+2q)} &= 0 \\ Q^{(+3q)}\psi^{(-3q)} &= \mathcal{F}^{(0)} \\ Q^{(+3q)}\mathcal{F}^{(0)} &= \nabla_a^{(+2q)}\psi^{a(+q)} = 0\end{aligned}\tag{3.43}$$

where we used

$$\mathcal{F}^{(0)} = \partial_a^{(+2q)} \mathcal{G}^{a(-2q)} - \partial^{a(-2q)} \mathcal{G}_a^{(+2q)} + ig_{YM} [\mathcal{G}_a^{(+2q)}, \mathcal{G}^{a(-2q)}] \quad (3.44)$$

- *Off shell case*

A set of off shell degrees of freedom is as in eqs(3.20-3.26); the supersymmetric transformations of the fields are therefore given by

$$\begin{aligned} Q^{(+3q)} \psi^{(-3q)} &= F^{(0)} \\ Q^{(+3q)} F^{(0)} &= 0 \\ Q^{(+3q)} \phi_{ab}^{(+q)} &= \mathcal{F}_{ab}^{(+4q)} \\ Q^{(+3q)} \mathcal{F}_{ab}^{(+4q)} &= 0 \end{aligned} \quad (3.45)$$

where $F^{(0)}$ is an auxiliary field; and

$$\begin{aligned} Q^{(+3q)} \mathcal{F}^{ab(-4q)} &= \psi^{ab(-q)} \\ Q^{(+3q)} \psi^{ab(-q)} &= 0 \\ Q^{(+3q)} \mathcal{F}^{(0)} &= \nabla_a^{(+2q)} \psi^{a(+q)} \\ Q^{(+3q)} \nabla_a^{(+2q)} \psi^{a(+q)} &= 0 \end{aligned} \quad (3.46)$$

where we have set

$$\psi^{ab(-q)} = \nabla^{a(-2q)} \psi^{b(+q)} - \nabla^{b(-2q)} \psi^{a(+q)} \quad (3.47)$$

4 Action in superspace

The action of twisted fields of chiral $3D \mathcal{N} = 4$ supersymmetry, exhibiting manifestly the scalar supercharge $Q^{(+3q)}$, reads in superspace like

$$\mathbf{L}_{twist} = \left(\int d\theta^{(-3q)} \mathcal{L}^{(-3q)} \right)_{\vartheta=0}, \quad (4.1)$$

with lagrangian superdensity $\mathcal{L}^{(-3q)}$ depending on the Grassman variable $\theta^{(-3q)}$; but also on

$$\vartheta^{a(+q)}, \quad (4.2)$$

Because of the role played by the supersymmetric generator $\mathcal{D}_a^{(-q)}$ in our construction; eg (3.21-3.24) and appendix, the dependence into the $\vartheta^{a(+q)}$ is implicit; and is killed at the end after performing integration with respect to $\theta^{(-3q)}$.

4.1 Lagrangian superdensity

The general form of the fermionic superdensity $\mathcal{L}^{(-3q)}$ scaling as $(mass)^{\frac{5}{2}}$ one may construct out of the set of gauge covariant superfields (3.20) is as follows

$$\begin{aligned}
\mathcal{L}^{(-3q)} = & \alpha_1 Tr \left[\Psi^{(-3q)} D^{(+3q)} \Psi^{(-3q)} \right] + \alpha_2 Tr \left[\Psi^{(-3q)} \mathbb{J}^{(0)} \right] + \\
& \alpha_3 Tr \left[\varepsilon_{abc} \Psi^{a(+q)} \mathbb{E}^{bc(-4q)} \right] + \\
& \alpha_4 Tr \left[\Phi_{ab}^{(+q)} \mathbb{E}^{ab(-4q)} \right] + \\
& \nu_{FI} Tr \left[\Psi^{(-3q)} \right]
\end{aligned} \tag{4.3}$$

where the α_i 's are normalization numbers and the coupling scaling as $(mass)^{\frac{3}{2}}$

$$\nu_{FI} \tag{4.4}$$

is a Fayet-Iliopoulos coupling constant. This term breaks scalar supersymmetry; it will be dropped out below.

Notice that the integration of with respect to the Grassmann variable of

$$\nu_{FI} \int d\theta^{(-3q)} Tr \left[\Psi^{(-3q)} \right] \tag{4.5}$$

leads in general to

$$\nu_{FI} Tr \left(F^{(0)} \right) = \nu_{FI} \sum_{A=1}^{\dim U(N)} F_A^{(0)} Tr \left(\mathcal{T}^A \right) \tag{4.6}$$

which does't vanish due to the abelian gauge subsymmetry

$$U(1) = \frac{U(N)}{SU(N)} \tag{4.7}$$

In the above relation, the matrices \mathcal{T}^A stand for the generators of $U(N)$.

4.2 Component field action

Using the θ - expansions (3.26-3.35) and integrating with respect to the Grassman variable $\theta^{(-3q)}$; we obtain

$$\begin{aligned}
\mathbf{L}_{twist} = & \alpha_1 Tr [F^{(0)} F^{(0)}] + \alpha_2 Tr [F^{(0)} \mathcal{F}^{(0)}] \\
& - \alpha_2 Tr [\psi^{(-3q)} \nabla_a^{(+2q)} \psi^{a(+q)}] \\
& + 2\alpha_3 Tr [\varepsilon_{abc} \psi^{a(+q)} \nabla^{b(-2q)} \psi^{c(+q)}] \\
& + \alpha_4 Tr [\mathcal{F}_{ab}^{(+4q)} \mathcal{F}^{ab(-4q)}] \\
& - \alpha_4 Tr [\phi_{ab}^{(+q)} [\nabla^{a(-2q)} \psi^{b(+q)} - \nabla^{b(-2q)} \psi^{a(+q)}]]
\end{aligned} \tag{4.8}$$

Notice that the terms with coefficients α_1 , α_2 , α_4 are manifestly invariant with respect to the scalar supersymmetric transformations. However the variation of the term

$$Tr [\varepsilon_{abc} \psi^{a(+q)} \nabla^{b(-2q)} \psi^{c(+q)}] \tag{4.9}$$

leads to

$$Tr [\varepsilon_{abc} \left\{ \psi^{a(+q)}, \left\{ \psi^{b(+q)}, \psi^{c(+q)} \right\} \right\}] \tag{4.10}$$

or equivalently

$$\frac{1}{3} \varepsilon_{abc} \psi_A^{a(+q)} \left(\nabla^{b(-2q)} \psi^{c(+q)} \right)_B \psi_C^{c(+q)} Tr ([\mathcal{T}^A, [\mathcal{T}^B, \mathcal{T}^C]] + \text{cyclic perm})$$

which vanishes identically due to Jacobi-Identity.

Notice that the component field action (4.8) can be rewritten in a more convenient form by eliminating $\phi_{ab}^{(+q)}$ through the constraint relation

$$\nabla_b^{(+2q)} \psi^{(-3q)} = \nabla^{a(-2q)} \phi_{ab}^{(+q)} \tag{4.11}$$

following from the superfield constraint eqs(3.25-3.39) namely

$$L_b^{(+2q)} \Psi^{(-3q)} = L^{a(-2q)} \Phi_{ab}^{(+q)} \tag{4.12}$$

Indeed, starting from

$$\begin{aligned} Tr \left(\phi_{ab}^{(+q)} \nabla^{a(-2q)} \psi^{b(+q)} \right) &= Tr \left(\phi_{ab}^{(+q)} \partial^{a(-2q)} \psi^{b(+q)} \right) \\ &\quad ig_{YM} Tr \left(\phi_{ab}^{(+q)} \left[\mathcal{G}^{a(-2q)}, \psi^{b(+q)} \right] \right) \end{aligned} \quad (4.13)$$

and integrating by part, we get up to a total divergence,

$$\begin{aligned} Tr \left(\phi_{ab}^{(+q)} \nabla^{a(-2q)} \psi^{b(+q)} \right) &= -Tr \left[\left(\nabla^{a(-2q)} \phi_{ab}^{(+q)} \right) \psi^{b(+q)} \right] \\ &= -Tr \left[\psi^{b(+q)} \nabla_b^{(+2q)} \psi^{(-3q)} \right] \end{aligned} \quad (4.14)$$

By substituting back into (4.8), we end with the lagrangian density

$$\begin{aligned} \mathbf{L}_{twist} &= \alpha_1 Tr \left[F^{(0)} F^{(0)} \right] + \alpha_2 Tr \left[F^{(0)} \mathcal{F}^{(0)} \right] \\ &\quad + \alpha_4 Tr \left[\mathcal{F}_{ab}^{(+4q)} \mathcal{F}^{ab(-4q)} \right] \\ &\quad + (\alpha_2 + 2\alpha_4) Tr \left[\psi^{a(+q)} \nabla_a^{(+2q)} \psi^{(-3q)} \right] \\ &\quad + 2\alpha_3 Tr \left[\varepsilon_{abc} \psi^{a(+q)} \nabla^{b(-2q)} \psi^{c(+q)} \right] \end{aligned} \quad (4.15)$$

Eliminating the auxiliary field $F^{(0)}$ through its equation of motion

$$F^{(0)} = -\frac{\alpha_2}{2\alpha_1} \mathcal{F}^{(0)} \quad (4.16)$$

and putting back into the lagrangian density, we end with

$$\begin{aligned} \mathbf{L}_{twist} &= \alpha_4 Tr \left[\mathcal{F}_{ab}^{(+4q)} \mathcal{F}^{ab(-4q)} \right] \\ &\quad - \frac{(\alpha_2)^2}{4\alpha_1} Tr \left[\mathcal{F}^{(0)} \mathcal{F}^{(0)} \right] \\ &\quad + (\alpha_2 + 2\alpha_4) Tr \left[\psi^{a(+q)} \nabla_a^{(+2q)} \psi^{(-3q)} \right] \\ &\quad + 2\alpha_3 Tr \left[\varepsilon_{abc} \psi^{a(+q)} \nabla^{b(-2q)} \psi^{c(+q)} \right] \end{aligned} \quad (4.17)$$

with scaling mass dimension $(mass)^3$. Notice that the YM coupling constant g_{YM} is within the gauge covariant derivatives $\nabla_a^{(+2q)}$, $\nabla^{a(-2q)}$ and the field strengths $\mathcal{F}_{ab}^{(+4q)}$, $\mathcal{F}^{ab(-4q)}$ as shown on eqs(3.29-3.30).

5 Twisted 3D $\mathcal{N} = 4$ SYM theory on lattice

In all what follows, we focuss on the study of twisted 3D $\mathcal{N} = 4$ supersymmetric YM on particular 3- dimensional lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$. This crystal is given by the fibration

$$\begin{array}{ccc} \mathcal{L}_{1D}^{u_1} & \rightarrow & \mathcal{L}_{3D}^{su_3 \times u_1} \\ & & \downarrow \\ & & \mathcal{L}_{2D}^{su_3} = \mathbb{A}_2^* \end{array} \quad (5.1)$$

with the two following components:

- (i) the base sublattice $\mathcal{L}_{2D}^{su_3}$ given by \mathbb{A}_2^* , the dual of the 2-dimensional root lattice \mathbb{A}_2 that is associated with the $SU(3)$ symmetry [33, 34]; and
- (ii) the fiber $\mathcal{L}_{1D}^{u_1}$ associated with the $U(1)$ factor of the symmetry $SU(3) \times U(1)$, it is a 1-dimensional lattice with direction normal to \mathbb{A}_2^* .

To fix the ideas, $\mathcal{L}_{3D}^{su_3 \times u_1}$ will be realized as *a twist* of the 3- dimensional lattice \mathbb{A}_3^* ; the dual to the 3D root lattice \mathbb{A}_3 generated by the 3 simple roots of $SU(4)$; that is:

$$\mathcal{L}_{3D}^{su_3 \times u_1} \sim \text{twist of } \mathbb{A}_3^* \quad (5.2)$$

Notice that the \mathbb{A}_3^* crystal is generated by the 3 fundamental weight of $SO(6) \simeq SU(4)$; and the twist of \mathbb{A}_3^* we are looking for is the one induced by the breaking mode

$$SU(4) \rightarrow SU(3) \times U(1) \quad (5.3)$$

For the explicit engineering of $\mathcal{L}_{3D}^{su_3 \times u_1}$; see next section; in due time let us focus on building the lattice analogue of the twist field of continuum.

5.1 Tensor fields on lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$

In this subsection, we study the discretization of the twisted 3D $\mathcal{N} = 4$ SYM theory to the lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$ of (5.1) by first focussing on the projection of this gauge theory on the base sublattice

$$\mathcal{L}_{2D}^{su_3} = \mathbb{A}_2^*$$

The implementation of the effect of the fiber $\mathcal{L}_{1D}^{u_1}$ will be considered later on.

5.1.1 Discretizing continuum

We begin by considering real scalar fields and then real tensor ones living on 3D space. After that, we give the extension to complex space and complex fields appearing in the formulation 3D $\mathcal{N} = 4$ SYM theory given in previous sections.

1) Discretizing space

In the discretization of the real 3D continuum space, generic points \mathbf{P} with local coordinates $(x^\mu) = (x, y, z)$ get mapped to lattice nodes

$$\mathbf{N} = \mathbf{N}(n_1, n_2, n_3)$$

described by 3- dimensional integral position vectors

$$\vec{R}_n$$

In the example of a *simple cubic* lattice with spacing parameter L , the nodes \mathbf{N} are represented by

$$\vec{R}_n = x_n \vec{e}_1 + y_n \vec{e}_2 + z_n \vec{e}_3 \quad (5.4)$$

with \vec{e}_i the usual canonical basis obeying

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$$

Each site \vec{R}_n in this simple lattice has 6 first nearest neighbors located at

$$\vec{R}_n \pm L\vec{e}_1, \quad \vec{R}_n \pm L\vec{e}_2, \quad \vec{R}_n \pm L\vec{e}_3 \quad (5.5)$$

In the case of the lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$ with an $SU(3) \times U(1)$ symmetry, the site positions \vec{R}_n are given by

$$\begin{aligned} \vec{R}_n &= n_1 \vec{L}_1 + n_2 \vec{L}_2 + n_3 \vec{L}_3 \\ n &= (n_1, n_2, n_3) \end{aligned} \quad (5.6)$$

with the basis \vec{L}_i satisfying a non trivial 3×3 intersection matrix

$$\begin{aligned}\mathcal{J}_{ij}^{su_3 \times u_1} &= \vec{L}_i \cdot \vec{L}_j \\ \vec{R}_n \cdot \vec{R}_m &= n_i \mathcal{J}_{ij}^{su_3 \times u_1} m_j\end{aligned}\tag{5.7}$$

capturing the shape of the crystal $\mathcal{L}_{3D}^{su_3 \times u_1}$.

The intersection matrix $\mathcal{J}_{ij}^{su_3 \times u_1}$ has a set of features; in particular the 2 following useful ones.

(a) the matrix $\mathcal{J}_{ij}^{su_3 \times u_1}$ is exactly given by

$$\mathcal{J}_{ij}^{su_3 \times u_1} = \begin{pmatrix} \frac{2}{3} + q^2 & \frac{1}{3} + 2q^2 & 3q^2 \\ \frac{1}{3} + 2q^2 & \frac{2}{3} + 4q^2 & 6q^2 \\ 3q^2 & 6q^2 & 9q^2 \end{pmatrix}\tag{5.8}$$

It depends on the number q that encodes the charges of the twisted supersymmetric YM fields under the abelian $U(1)$ symmetry of (5.3) and moreover defines the fibration (2.33).

(b) In the particular and remarkable case $q = 0$, the intersection matrix $\mathcal{J}_{ij}^{su_3 \times u_1}$ reduces to the singular matrix

$$(\mathcal{J}_{ij}^{su_3 \times u_1})_{q=0} = \begin{pmatrix} \mathcal{J}_{ij}^{su_3} & 0 \\ 0 & 0 \end{pmatrix}\tag{5.9}$$

with

$$\mathcal{J}_{ij}^{su_3} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\tag{5.10}$$

This singular case corresponds exactly to the projection of sites

$$\vec{R}_{(n_1, n_2, n_3)}\tag{5.11}$$

in the 3- dimensional $\mathcal{L}_{3D}^{su_3 \times u_1}$ onto sites

$$\vec{r}_{(n_1, n_2)}\tag{5.12}$$

in the base sublattice

$$\mathbb{A}_2^*$$

This means that site positions in $\mathcal{L}_{3D}^{su_3 \times u_1}$ depend on the parameter q ; and so can be parameterized like

$$\vec{R}_{(n_1, n_2, n_3)}^{(q)} = \begin{pmatrix} \vec{r}_{(n_1, n_2)} \\ qZ_{n_3} \end{pmatrix} \quad (5.13)$$

with third component belonging to the fiber,

$$Z_{n_3} \in \mathcal{L}_{1D}^{u_1} \quad , \quad \mathcal{L}_{1D}^{u_1} \simeq q\mathbb{Z} \quad (5.14)$$

The same property is valid for the \vec{L}_i basis generators; they depend on the charge q and may be decomposed as well like

$$\vec{L}_1^{(q)} = \begin{pmatrix} \vec{l}_1 \\ q \end{pmatrix}, \quad \vec{L}_2^{(q)} = \begin{pmatrix} \vec{l}_2 \\ 2q \end{pmatrix}, \quad \vec{L}_3^{(q)} = \begin{pmatrix} \vec{0} \\ 3q \end{pmatrix} \quad (5.15)$$

with the 2-dimensional vectors \vec{r}_n giving the sites in the base sublattice \mathbb{A}_2^*

$$\begin{aligned} \vec{r}_n &= n_1 \vec{l}_1 + n_2 \vec{l}_2 \\ n &= (n_1, n_2) \end{aligned} \quad (5.16)$$

So the case $q = 0$ define a projection from $\mathcal{L}_{3D}^{su_3 \times u_1}$ down to the base \mathbb{A}_2^* ; we have

$$\vec{R}_n^{(0)} = \begin{pmatrix} \vec{r}_n \\ 0 \end{pmatrix} \quad (5.17)$$

2) Discrete field variables

- *scalar fields*

Under discretization of the real $3D$ continuum space into $\mathcal{L}_{3D}^{su_3 \times u_1}$, local *scalar* fields $\Phi(x)$ of the continuum get mapped to an infinite set of discrete variables

$$\begin{aligned} \Phi(R_n) &= \Phi_{(n_1, n_2, n_3)}^{(q)} \\ &\equiv \Phi_n^{(q)} \end{aligned} \quad (5.18)$$

living at the lattice nodes

$$R_n, \quad n = (n_1, n_2, n_3) \in \mathbb{Z}^3$$

The variables $\Phi_n^{(q)}$ may have either an even statistics or an odd one depending on whether $\Phi(x)$ is bosonic or fermionic. In the case of twisted $3D \mathcal{N} = 4$ supersymmetric YM, odd variables are given by the twisted fermion $\psi^{(-3q)}$ and the Grassman variable $\theta^{(-3q)}$.

- *antisymmetric p-tensors*

Real p- form fields $\mathcal{T}_{[p]}(x)$ in continuum

$$\mathcal{T}_{[p]} = \frac{1}{p!} dx^{\mu_1} \wedge dx^{\mu_2} \dots \wedge dx^{\mu_p} T_{\mu_1 \dots \mu_p} \quad (5.19)$$

are associated with p- dimensional plaquettes in the lattice. In the case of $\mathcal{L}_{3D}^{su_3 \times u_1}$, we have vectors and their duals namely the rank 2 antisymmetric tensors; rank 3 antisymmetric tensors are dual to scalars. So we have

fields	→	p-plaquettes	
T		sites	
T_μ		1d- <i>links</i>	(5.20)
$T_{\mu\nu}$		2d- plaquettes	~ 1d- <i>links</i>

Let us illustrate the construction on the particular case of the gradient $\partial_\mu \Phi(x)$. To get the discrete expression representing $\partial_\mu \Phi(x)$ on the lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$, it is useful to consider

$$d\Phi(x) = dx^\mu \partial_\mu \Phi(x) \quad (5.21)$$

which is a particular *1-form* in 3D. This differential, which involves the operator $d = dx^\mu \partial_\mu$, behaves as a scalar under $SO(3)$ and is related to $\Phi(x)$ by

$$d\Phi(\vec{x}) = \lim_{d\vec{x} \rightarrow 0} [\Phi(\vec{x} + d\vec{x}) - \Phi(\vec{x})] \quad (5.22)$$

In the standard case of a simple cubic lattice with spacing parameter L , the arbitrary elementary variations $d\vec{x}$ of the 3-dimensional continuum are given by the 6 first nearest neighbors namely

$$\pm L\vec{e}_1, \quad \pm L\vec{e}_2, \quad \pm L\vec{e}_3 \quad (5.23)$$

In the case of discretization of space to the $\mathcal{L}_{3D}^{su_3 \times u_1}$ of eq(5.3), vectors \vec{x} in continuum are mapped to $\vec{R}_n^{(q)}$ and the elementary variations $d\vec{x}$ are mapped to first nearest neighbors of $\vec{R}_n^{(q)}$; that is

$$\vec{x} + d\vec{x} \rightarrow \vec{R}_n^{(q)} + \vec{V}_I^{(q)} \quad (5.24)$$

with

$$\vec{V}_I^{(q)} = \begin{pmatrix} \vec{v}_I \\ qZ_I \end{pmatrix} \quad (5.25)$$

and the 6 non zero \vec{v}_I 's as

$$\vec{v}_i = \begin{pmatrix} v_i^1 \\ v_i^2 \end{pmatrix}, \quad i = 1, \dots, 6 \quad (5.26)$$

referring to the first nearest neighbors of the base sublattice \mathbb{A}_2^* . So the 3D crystal analogue of $d\Phi(x)$ is given by

$$\Phi_I(R_n) = \Phi(R_n + V_I^{(q)}) - \Phi(R_n) \quad (5.27)$$

with projection on the \mathbb{A}_2^* base sublattice corresponding to $q = 0$ as follows

$$\phi_I(\vec{r}_n) = \phi(\vec{r}_n + \vec{v}_I) - \phi(\vec{r}_n) \quad (5.28)$$

where the \vec{v}_I 's are as in (5.25). For later use, it is convenient to denote $\phi_I(\vec{r}_n)$ like

$$\phi_I(\vec{r}_n) = \phi_{\mathbf{r}_n \rightarrow (\mathbf{r}_n + \mathbf{v}_I)} \equiv \phi_{n,I} \quad (5.29)$$

5.1.2 *First result*

From eqs(5.27-5.28), we learn a set of useful features that we collect below:

- the field $\partial_\mu \Phi(x)$ is mapped to link variables

$$\Phi_{n,I}^{(q)} \quad (5.30)$$

living on edges of the 3D crystal $\mathcal{L}_{3D}^{su_3 \times u_1}$.

For $q = 0$, the link variables $\Phi_{n,I}^{(q)}$ are projected down to

$$\phi_{n,I} \quad (5.31)$$

living on the \mathbb{A}_2^* the edge

$$\overrightarrow{P_{\mathbf{r}_n} P_{\mathbf{r}_n + \mathbf{v}_I}} \sim \tilde{\mathbf{v}}_I \quad (5.32)$$

- the analogue of the gauge field $A_\mu(x)$ on the lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$ is given by the link variables

$$\mathcal{U}_I(R_n^{(q)}) = \mathcal{U}_{n,I}^{(q)} \quad (5.33)$$

with projection on the \mathbb{A}_2^* base sublattice as

$$\mathcal{U}_I(\vec{r}_n) = U_{n,I} \quad (5.34)$$

living on the links (5.32).

The usual gauge transformation in continuum with G a generic gauge group element

$$\partial_\mu + ig_{YM} A_\mu(x) \rightarrow G(x) [\partial_\mu + ig_{YM} A_\mu(x)] G^\dagger(x) \quad (5.35)$$

is mapped to

$$\mathcal{U}_I(R_n^{(q)}) \rightarrow G(R_n^{(q)}) \mathcal{U}_I(R_n^{(q)}) G^\dagger(R_n^{(q)} + V_I) \quad (5.36)$$

On the \mathbb{A}_2^* base sublattice, these transformations reduce to

$$U_I(\vec{r}_n) \rightarrow G(\vec{r}_n) U_I(\vec{r}_n) G^\dagger(\vec{r}_n + \vec{v}_I) \quad (5.37)$$

- the discrete analogue of the field strength $F_{\mu\nu}(x)$ is given by the plaquette variables

$$\mathcal{W}_{[IJ]}(R_n^{(q)}) = \mathcal{W}_{n,[IJ]}^{(q)}, \quad I \neq J \quad (5.38)$$

which is dual to 1-dimensional link. On the \mathbb{A}_2^* base sublattice, these variables are projected to the 2- dimensional plaquette variables

$$W_{[IJ]}(\mathbf{r}_n) = W_{n,[IJ]}, \quad I \neq J \quad (5.39)$$

associated with

$$\overrightarrow{P_{\mathbf{r}_n}} \overrightarrow{P_{\mathbf{r}_n + \mathbf{v}_I}} \wedge \overrightarrow{P_{\mathbf{r}_n}} \overrightarrow{P_{\mathbf{r}_n + \mathbf{v}_J}} \sim \tilde{\mathbf{v}}_I \wedge \tilde{\mathbf{v}}_J \quad (5.40)$$

This plaquette has 4 vertices located at

$$\begin{aligned} P_{\mathbf{r}_n} &\sim \vec{r}_n, & P_{\mathbf{r}_n + \mathbf{v}_I} &\sim \vec{r}_n + \vec{v}_I \\ P_{\mathbf{r}_n + \mathbf{v}_J} &\sim \vec{r}_n + \vec{v}_J, & P_{\mathbf{r}_n + \mathbf{v}_I + \mathbf{v}_J} &\sim \vec{r}_n + \vec{v}_I + \vec{v}_J \end{aligned} \quad (5.41)$$

and has an interpretation in terms of the vector surface $\vec{s}_{IJ} = \vec{v}_I \wedge \vec{v}_J$ with components

$$s_{IJ}^\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho} v_I^\nu v_J^\rho \quad (5.42)$$

- Non abelian $U(N)$ gauge fields

In the case of YM theory with non abelian $U(N)$ gauge symmetry, the fields are valued in the adjoint representation of the Lie algebra of the gauge symmetry; so the gradient $\partial_\mu \Phi(x)$ and the field strength $F_{\mu\nu}$ involve gauge covariant derivatives

$$\begin{aligned} D_\mu \Phi &= \partial_\mu \Phi + [A_\mu, \Phi] \\ F_{\mu\nu} &= [D_\mu, D_\nu] \end{aligned} \quad (5.43)$$

By discretization, $D_\mu \Phi(x)$ and $F_{\mu\nu}(x)$ are respectively mapped to discrete $N \times N$ matrix variables

$$\Phi_I^{(q)}(R_n) \quad , \quad \mathcal{W}_{IJ}^{(p)}(\vec{r}_n) \quad (5.44)$$

carrying moreover charges under $U(1)$. On the base sublattice \mathbb{A}_2^* where $q = 0$, these quantities become

$$\begin{aligned} \Phi_I(\vec{r}_n) &= U_I(\vec{r}_n) \Phi(\vec{r}_n + \vec{v}_I) - \Phi(\vec{r}_n) U_I(\vec{r}_n) \\ W_{IJ}(\vec{r}_n) &= U_I(\vec{r}_n) U_J(\vec{r}_n + \vec{v}_I) - U_J(\vec{r}_n) U_I(\vec{r}_n + \vec{v}_J) \end{aligned} \quad (5.45)$$

5.2 Complex extension and orientation

In the case of the complex 3D space, on which the 3-dimensional twisted $\mathcal{N} = 4$ supersymmetric YM has been formulated, one distinguishes two kinds of quantities:

- complex antisymmetric tensor fields of type $B^{(q)a_1 \dots a_p}$ transforming in some complex representation \mathbf{R}_q of $SU(3) \times U(1)$,
- the adjoint fields $B_{a_1 \dots a_p}^{(-q)}$ transforming in the adjoint conjugate $\bar{\mathbf{R}}_{-q}$.

On the lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$, these objects are interpreted in terms of *oriented* p-simplex. To that purpose, recall the objects appearing in the field action (4.8-4.17); there, we have bosons and fermions: In the bosonic sector, each object have an adjoint as shown below

Object	Adjoint	
$z^{a(-2q)}$	$\bar{z}_a^{(+2q)}$	
$dz^{a(-2q)}$	$d\bar{z}_a^{(+2q)}$	
$\mathcal{G}^{a(-2q)}$	$\bar{\mathcal{G}}_a^{(+2q)}$	
$\nabla^{a(-2q)}$	$\nabla_a^{(+2q)}$	
$\mathcal{F}^{ab(-4q)}$	$\bar{\mathcal{F}}_{ab}^{(+4q)}$	

(5.46)

and so both orientations of bosonic link variables are involved; contrary to the chiral fermionic sector

Object	Adjoint
$\theta^{(-3q)}$	-
$\vartheta^{a(+q)}$	-
$\psi^{(-3q)}$	-
$\psi^{a(+)}$	-
$\nabla^{a(-2q)}\psi^{b(+)}$	-
$\nabla_a^{(+2q)}\psi^{a(+)}$	-

(5.47)

where we have only one orientation for fermionic lattice link variables.

To study the discrete version of (4.8-4.17), we proceed in two steps:

a) step 1: we describe lattice theory living on the 2-dimensional \mathbb{A}_2^* given by fig 2.

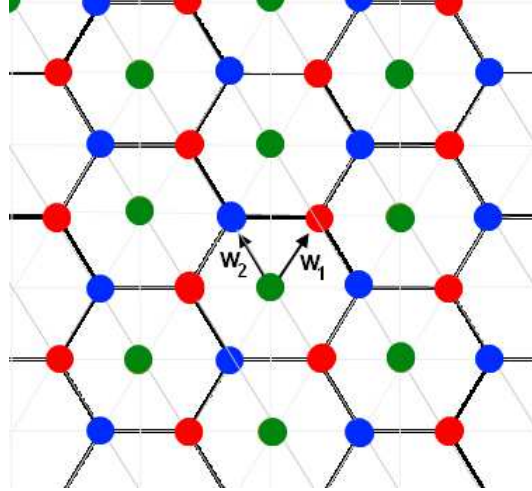


Figure 2: the lattice \mathbb{A}_2^* generated by the 2 basic weight vectors \vec{w}_1, \vec{w}_2 of $SU(3)$. Green nodes are associated with the lattice variable $\psi_n^{(-3q)}$; red nodes with $\psi_n^{I(+q)}$ and blue nodes with the lattice gauge variables $(U_n^{I(-2q)}, U_{In}^{(+2q)})$. More precisely $\psi_n^{I(+q)}$ transforms into the representation $\mathbf{3}$; it is given by the link from Green to red nodes. Similarly, $U_n^{I(-2q)}$; it transforms in the $\mathbf{3}$ representation and is given by links from the blue to the green nodes while $U_{In}^{(+2q)}$ transforms in the adjoint $\bar{\mathbf{3}}$ and is given by links from the green to the blue sites.

b) step 2: we extend the construction from \mathbb{A}_2^* to the 3-dimensional lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$ by implementing the fiber direction $\mathcal{L}_{1D}^{u_1}$. This corresponds to unfolding the normal direction to \mathbb{A}_2^* as illustrated on fig 3.

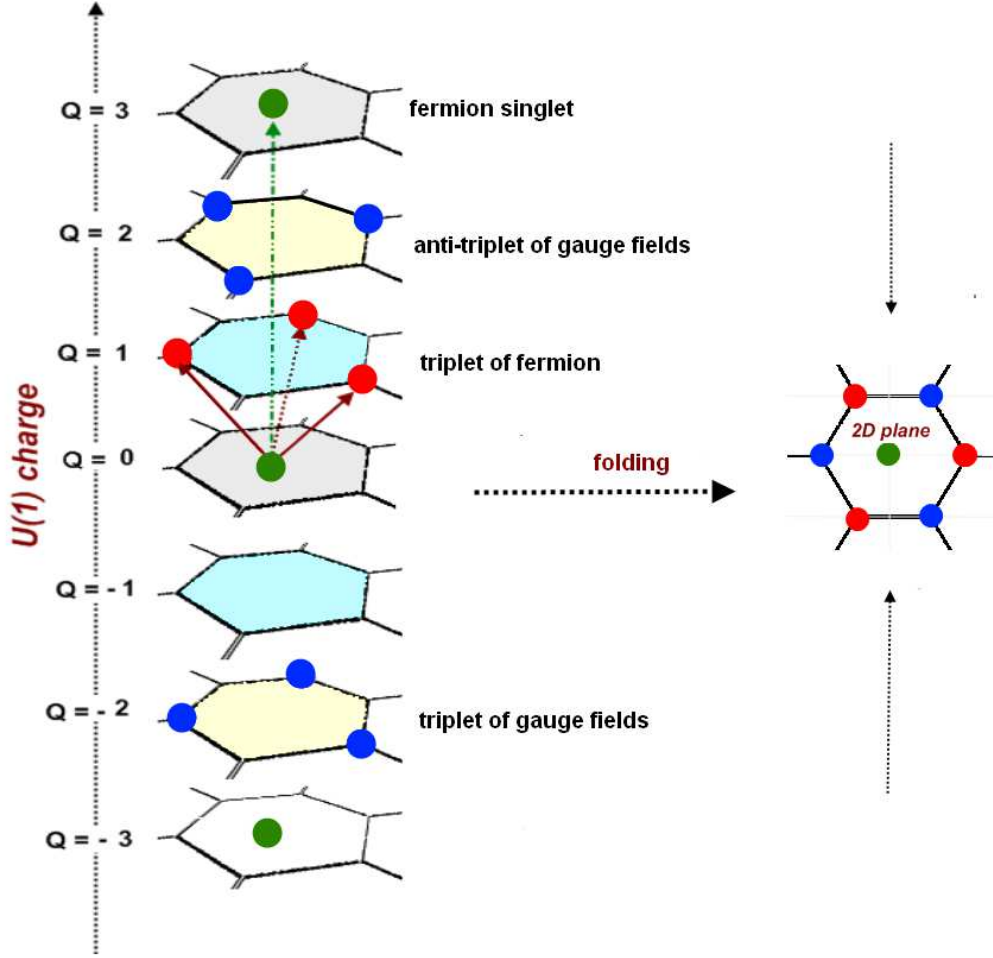


Figure 3: the 3D lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$ given by a $\mathcal{L}_{2D}^{su_3} \times \mathcal{L}_{1D}^{u_1}$ fibration with 2D base sublattice given by \mathbb{A}_2^* and 1D fiber $\mathcal{L}_{1D}^{u_1}$ isomorphic to \mathbb{Z} , the set of integers. The superposition of the 3 sublattices \mathfrak{G} , \mathfrak{R} and \mathfrak{B} making \mathbb{A}_2^* as given by eq(5.57) is lifted. Sheets with $Q = 3q$ corresponds to the sublattice \mathfrak{G} , sheets with $Q = q$ to the sublattice \mathfrak{R} and those with $Q = 2q$ to the sublattice \mathfrak{B} .

5.2.1 *twisted supersymmetric YM on \mathbb{A}_2^**

The lattice \mathbb{A}_2^* is generated by the 2 fundamental weight vectors $\vec{\omega}_1, \vec{\omega}_2$ of the Lie algebra of the simple $SU(3)$ symmetry. These are non canonical vectors obeying

$$\vec{\omega}_i \cdot \vec{\omega}_i = \frac{2}{3} \quad , \quad \widehat{(\vec{\omega}_1, \vec{\omega}_2)} = \frac{\pi}{6} \quad (5.48)$$

Each site \vec{r}_n in \mathbb{A}_2^* (say a green site of fig 2) has 6 first nearest neighbors located at $\vec{r}_n + \vec{v}_i$ and which can be organized into 2 subsets, each having 3 elements like

$$\begin{aligned} \vec{v}_1 &= L\sqrt{\frac{3}{2}}\vec{\omega}_1 \\ \vec{v}_2 &= L\sqrt{\frac{3}{2}}(\vec{\omega}_2 - \vec{\omega}_1) \\ \vec{v}_3 &= -L\sqrt{\frac{3}{2}}\vec{\omega}_2 \end{aligned} \quad (5.49)$$

and

$$\begin{aligned} \vec{v}'_1 &= -\vec{v}_1 \\ \vec{v}'_2 &= -\vec{v}_2 \\ \vec{v}'_3 &= -\vec{v}_3 \end{aligned} \quad (5.50)$$

Notice that each triplet obeys a traceless property

$$\begin{aligned} \vec{v}_1 + \vec{v}_2 + \vec{v}_3 &= \vec{0} \\ \vec{v}'_1 + \vec{v}'_2 + \vec{v}'_3 &= \vec{0} \end{aligned} \quad (5.51)$$

Notice also that the set \mathfrak{G} of (green) sites \vec{r}_n at the centre of the hexagons of fig 2 form a sublattice of \mathbb{A}_2^* generated by the particular vectors

$$\begin{aligned} \vec{\alpha}_1 &= 2\vec{\omega}_1 - \vec{\omega}_2 \\ \vec{\alpha}_2 &= \vec{\omega}_1 - 2\vec{\omega}_2 \end{aligned} \quad (5.52)$$

This means that green sites are related amongst others as

$$\vec{r}_m = \sum_{m_1, m_2} m_1 \vec{\alpha}_1 + m_2 \vec{\alpha}_2 \quad , \quad (5.53)$$

and form a sublattice

$$\mathfrak{G} \equiv \{\vec{r}_m\}_{\text{green sites}}$$

which is nothing but \mathbb{A}_2 , the *root lattice* of $SU(3)$. So we have the isomorphism

$$\mathfrak{G} \simeq \mathbb{A}_2 \quad (5.54)$$

Notice moreover that, because of the symmetric role played by the 3 types of nodes (green, red, blue in fig 2), the same thing may be said about the set \mathfrak{R} of red sites and the set \mathfrak{B} of blue ones. In other words the set \mathfrak{R} is isomorphic to a root lattice of SU(3) and similarly the set \mathfrak{B} is isomorphic as well to a root lattice of SU(3). Thus we have the isomorphisms

$$\mathfrak{R} \simeq \mathbb{A}_2 \quad , \quad \mathfrak{B} \simeq \mathbb{A}_2 \quad (5.55)$$

and formally

$$\mathfrak{R} \simeq \mathfrak{G} + \vec{\omega}_1 \quad , \quad \mathfrak{B} \simeq \mathfrak{G} + \vec{\omega}_2 \quad (5.56)$$

From this representation, it follows that the lattice \mathbb{A}_2^* is made by the superposition of the 3 sublattices \mathfrak{G} , \mathfrak{R} and \mathfrak{B} or equivalently

$$\mathbb{A}_2^* = \mathfrak{G} \cup \mathfrak{R} \cup \mathfrak{B} \quad (5.57)$$

For more details on the matrix describing the shape of the 2- dimensional base lattice \mathbb{A}_2^* ; see subsection 5.

Under discretization of continuum, we have the following correspondence

continuum	crystal \mathbb{A}_2^*	
(z, \bar{z})	\vec{r}_n	
$z + dz$	$\vec{r}_n + \vec{v}_I$	(5.58)
$\bar{z} + d\bar{z}$	$\vec{r}_n - \vec{v}_I$	

with $n = (n_1, n_2)$ arbitrary integers; and where \vec{v}_I stand for the 3 oriented first nearest neighbors given by (5.49). Observe that complex conjugation is captured by the change of the orientation of \vec{v}_I .

Regarding the lattice analogue of the twist fields in continuum, we have the following dictionary:

(i) **bosonic fields** :

continuum	crystal A_2^*	
$\mathcal{G}^a(z, \bar{z})$	U_n^I	
$\bar{\mathcal{G}}_a(z, \bar{z})$	$U_{n,I}^\dagger$	
$\mathcal{F}^{ab}(z, \bar{z})$	W_n^{IJ}	(5.59)
$\bar{\mathcal{F}}_{ab}(z, \bar{z})$	$W_{n,IJ}^\dagger$	
$\mathcal{F}_a^b(z, \bar{z})$	$W_{n,I}^J$	

with

$$\begin{aligned}
U_n^I &= U^I(\vec{r}_n) \\
U_{n,I}^\dagger &= U_I^\dagger(\vec{r}_n) \\
W_n^{IJ} &= U^I(\vec{r}_n) U^J(\vec{r}_n + \vec{v}_I) - U^J(\vec{r}_n) U^I(\vec{r}_n + \vec{v}_J) \\
W_{n,IJ}^\dagger &= U_J^\dagger(\vec{r}_n + \vec{v}_I) U_I^\dagger(\vec{r}_n) - U_I^\dagger(\vec{r}_n + \vec{v}_J) U_J^\dagger(\vec{r}_n)
\end{aligned} \tag{5.60}$$

and

$$W_{n,I}^I = U^I(\vec{r}_n) U_I^\dagger(\vec{r}_n) - U_I^\dagger(\vec{r}_n - \vec{v}_I) U^I(\vec{r}_n - \vec{v}_I) \tag{5.61}$$

(ii) ***fermionic fields*** :

continuum	crystal A_2^*	
$\psi(z, \bar{z})$	ψ_n	
$\psi^a(z, \bar{z})$	ψ_n^I	
$\nabla^a \psi^b$	ψ_n^{IJ}	
$\bar{\nabla}_a \psi^a$	$\psi_{n,I}^I$	

(5.62)

with

$$\begin{aligned}
\psi_n &= \psi(\vec{r}_n) \\
\psi_n^I &= \psi^I(\vec{r}_n) \\
\psi_n^{IJ} &= \psi^{IJ}(\vec{r}_n) \\
\psi_{n,I}^I &= \psi_I^I(\vec{r}_n)
\end{aligned} \tag{5.63}$$

and

$$\begin{aligned}
\psi^{IJ}(\vec{r}_n) &= U^I(\vec{r}_n) \psi^J(\vec{r}_n + \vec{v}_I) - \psi^J(\vec{r}_n) U^I(\vec{r}_n + \vec{v}_J) \\
\psi_I^I(\vec{r}_n) &= \psi^I(\vec{r}_n) U_I^\dagger(\vec{r}_n) - U_I^\dagger(\vec{r}_n - \vec{v}_I) \psi^I(\vec{r}_n - \vec{v}_I)
\end{aligned} \tag{5.64}$$

The $U(N)$ gauge transformations with generic unitary $N \times N$ matrix $G(\vec{r}_n)$ act on the

lattice fields as follows

field	\rightarrow	gauge transform	
U_n^I		$G(\vec{r}_n) U^I(\vec{r}_n) G^\dagger(\vec{r}_n + \vec{v}_I)$	
$U_{n,I}^\dagger$		$G(\vec{r}_n + \vec{v}_I) U_I^\dagger(\vec{r}_n) G^\dagger(\vec{r}_n)$	
W_n^{IJ}		$G(\vec{r}_n) W_n^{IJ} G^\dagger(\vec{r}_n + \vec{v}_I + \vec{v}_J)$	
$W_{n,IJ}^\dagger$		$G(\vec{r}_n + \vec{v}_I + \vec{v}_J) W_n^{IJ} G^\dagger(\vec{r}_n)$	(5.65)
$\psi(\vec{r}_n)$		$G(\vec{r}_n) \psi(\vec{r}_n) G^\dagger(\vec{r}_n)$	
$\psi^I(\vec{r}_n)$		$G(\vec{r}_n) \psi^I(\vec{r}_n) G^\dagger(\vec{r}_n + \vec{v}_I)$	
$\psi^{IJ}(\vec{r}_n)$		$G(\vec{r}_n) \psi^{IJ}(\vec{r}_n) G^\dagger(\vec{r}_n + \vec{v}_I + \vec{v}_J)$	
$\psi_I^I(\vec{r}_n)$		$G(\vec{r}_n) \psi_I^I(\vec{r}_n) G^\dagger(\vec{r}_n)$	

(5.66)

Using these gauge transformations, one can check that the following couplings

$$\begin{aligned}
(i) \quad & : \quad Tr [\psi(\vec{r}_n) \psi_I^I(\vec{r}_n)] \\
(ii) \quad & : \quad -\frac{1}{2} \varepsilon_{IJK} Tr [\psi^K(\vec{r}_n - \vec{v}_K) \psi^{IJ}(\vec{r}_n)]
\end{aligned}
\tag{5.67}$$

are gauge invariant provided we have

$$G^\dagger(\vec{r}_n + \vec{v}_I + \vec{v}_J) G(\vec{r}_n - \vec{v}_K) = I \tag{5.68}$$

But this constraint equation requires

$$\vec{v}_I + \vec{v}_J + \vec{v}_K = \vec{0} \tag{5.69}$$

which, up on using the antisymmetry property of the tensor ε_{IJK} , can be also written as

$$\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0} \tag{5.70}$$

This constraint relation is identically satisfied for the lattice \mathbb{A}_2^* as shown by eq(5.51).

5.2.2 action of twisted supersymmetric YM on \mathbb{A}_2^*

Using the above dictionary giving the analogue of fields in continuum to lattice variables on $\mathcal{L}_{3D}^{su_3 \times u_1}$, we can work out the action of twisted supersymmetric YM on the base sublattice \mathbb{A}_2^* . This action may be decomposed in 2 parts as

$$\mathcal{S}_{lattice} = \mathcal{S}_{lattice}^{bose} + \mathcal{S}_{lattice}^{fermi} \quad (5.71)$$

where $\mathcal{S}_{lattice}^{bose}$ involving bosonic degrees of freedom and $\mathcal{S}_{lattice}^{fermi}$ describing lattice fermionic variables coupled to the gauge link variables.

(α) *Bosonic term*

Under discretization, the bosonic part of the field action in continuum namely

$$\begin{aligned} \mathcal{S}_{cont}^{bose} &= \alpha_4 \int Tr \left[\mathcal{F}_{ab}^{(+4q)} \mathcal{F}^{ab(-4q)} \right] \\ &\quad - \frac{(\alpha_2)^2}{4\alpha_1} \int Tr \left[\mathcal{F}^{(0)} \mathcal{F}^{(0)} \right] \end{aligned} \quad (5.72)$$

gets mapped to the following gauge invariant lattice one

$$\mathcal{S}_{latt}^{bose} = \alpha_4 \sum_{\mathbb{A}_2^*} Tr \left(W_n^{IJ} W_{n,IJ}^\dagger \right) - \frac{(\alpha_2)^2}{4\alpha_1} \sum_{\mathbb{A}_2^*} Tr \left(W_n^{(0)} W_n^{\dagger(0)} \right) \quad (5.73)$$

with

$$W_n^{IJ} W_{n,IJ}^\dagger = \mathcal{P}_1 + \mathcal{P}_2 - \mathcal{P}_3 - \mathcal{P}_4 \quad (5.74)$$

$$W_n^{(0)} W_n^{\dagger(0)} = \mathcal{R}_1 + \mathcal{R}_2 - \mathcal{R}_3 - \mathcal{R}_4$$

and

$$\begin{aligned} \mathcal{P}_1 &= U^I(\vec{r}_n) U^J(\vec{r}_n + \vec{v}_I) U_J^\dagger(\vec{r}_n + \vec{v}_I) U_I^\dagger(\vec{r}_n) \\ \mathcal{P}_2 &= U^J(\vec{r}_n) U^I(\vec{r}_n + \vec{v}_J) U_I^\dagger(\vec{r}_n + \vec{v}_J) U_J^\dagger(\vec{r}_n) \\ \mathcal{P}_3 &= U^J(\vec{r}_n) U^I(\vec{r}_n + \vec{v}_J) U_J^\dagger(\vec{r}_n + \vec{v}_I) U_I^\dagger(\vec{r}_n) \\ \mathcal{P}_4 &= U^I(\vec{r}_n) U^J(\vec{r}_n + \vec{v}_I) U_I^\dagger(\vec{r}_n + \vec{v}_J) U_J^\dagger(\vec{r}_n) \end{aligned} \quad (5.75)$$

as well as

$$\begin{aligned}
\mathcal{R}_1 &= U^I(\vec{r}_n) U_I^\dagger(\vec{r}_n) U^J(\vec{r}_n) U_J^\dagger(\vec{r}_n) \\
\mathcal{R}_2 &= U_I^\dagger(\vec{r}_n - \vec{v}_I) U^I(\vec{r}_n - \vec{v}_I) U_J^\dagger(\vec{r}_n - \vec{v}_J) U^J(\vec{r}_n - \vec{v}_J) \\
\mathcal{R}_3 &= U^I(\vec{r}_n) U_I^\dagger(\vec{r}_n) U_J^\dagger(\vec{r}_n - \vec{v}_J) U^J(\vec{r}_n - \vec{v}_J) \\
\mathcal{R}_4 &= U_I^\dagger(\vec{r}_n - \vec{v}_I) U^I(\vec{r}_n - \vec{v}_I) U^J(\vec{r}_n) U_J^\dagger(\vec{r}_n)
\end{aligned} \tag{5.76}$$

(β) *fermionic term*

For the fermionic terms, the analogue of

$$\begin{aligned}
\mathcal{S}_{cont}^{fermi} &= (\alpha_2 + 2\alpha_4) \int Tr \left[\psi^{a(+q)} \nabla_a^{(+2q)} \psi^{(-3q)} \right] \\
&+ 2\alpha_3 \int Tr \left[\varepsilon_{abc} \psi^{a(+q)} \nabla^{b(-2q)} \psi^{c(+q)} \right]
\end{aligned} \tag{5.77}$$

is given by the following gauge invariant expression

$$\begin{aligned}
\mathcal{S}_{latt}^{fermi} &= (\alpha_2 + 2\alpha_4) \sum_{\mathbb{A}_2^*} Tr \left[\psi(\vec{r}_n) \psi^I(\vec{r}_n) U_I^\dagger(\vec{r}_n) \right] + \\
&(\alpha_2 + 2\alpha_4) \sum_{\mathbb{A}_2^*} Tr \left[\psi(\vec{r}_n) U_I^\dagger(\vec{r}_n - \vec{v}_I) \psi^I(\vec{r}_n - \vec{v}_I) \right] \\
&+ 2\alpha_3 \sum_{\mathbb{A}_2^*} \varepsilon_{IJK} Tr \left[\psi^K(\vec{r}_n - \vec{v}_K) \psi^{IJ}(\vec{r}_n) \right]
\end{aligned} \tag{5.78}$$

with $\psi^{IJ}(\vec{r}_n)$ as in eq(5.64).

6 Twisted theory on the 3D lattice

First, we construct the 3D lattice by giving further details on the base sublattice \mathbb{A}_2^* ; and the fiber $\mathcal{L}_{1D}^{u_1}$. Then we turn to derive the gauge invariant lattice action of twisted supersymmetric YM theory on $\mathcal{L}_{3D}^{su_3 \times u_1}$.

6.1 More on the base sublattice \mathbb{A}_2^*

The base sublattice \mathbb{A}_2^* is generated by the 2 fundamental weight vectors $\vec{\omega}_1, \vec{\omega}_2$ of $SU(3)$. These fundamental weight vectors, having the length $\frac{2}{3}$ and angle $(\widehat{\omega_1, \omega_2}) = \frac{\pi}{6}$, are the dual of the 2 simple roots $\vec{\alpha}_1, \vec{\alpha}_2$ of $SU(3)$,

$$\vec{\omega}_i \cdot \vec{\alpha}_j = \delta_{ij} \quad (6.1)$$

Below, we take $\vec{\omega}_1, \vec{\omega}_2$ and $\vec{\alpha}_1, \vec{\alpha}_2$ in the real plane as follows

$$\begin{aligned} \vec{\omega}_1 &= \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{6} \right) \quad , \quad \vec{\omega}_2 = \left(0, \frac{\sqrt{6}}{3} \right) \\ \vec{\alpha}_1 &= (\sqrt{2}, 0) \quad , \quad \vec{\alpha}_2 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2} \right) \end{aligned} \quad (6.2)$$

They are related to each other like

$$\begin{aligned} \vec{\alpha}_1 &= 2\vec{\omega}_1 - \vec{\omega}_2 \\ \vec{\alpha}_2 &= 2\vec{\omega}_2 - \vec{\omega}_1 \end{aligned} \quad (6.3)$$

and

$$\begin{aligned} \vec{\omega}_1 &= \frac{1}{3} (2\vec{\alpha}_1 + \vec{\alpha}_2) \\ \vec{\omega}_2 &= \frac{1}{3} (\vec{\alpha}_1 + 2\vec{\alpha}_2) \end{aligned} \quad (6.4)$$

The vectors $\vec{\alpha}_1, \vec{\alpha}_2$ generate \mathbb{A}_2 ; the root lattice of $SU(3)$.

$$\begin{aligned} \vec{r}_n \in \mathbb{A}_2 \quad \Leftrightarrow \quad \vec{r}_n &= \sum_{n,m} n\vec{\alpha}_1 + m\vec{\alpha}_2 \\ &= \sum_{n,m} (2n - m)\vec{\omega}_1 + (2m - n)\vec{\omega}_2 \end{aligned}$$

Using $\vec{\omega}_1, \vec{\omega}_2$, position vectors \vec{r}_n of sites in the wight lattice \mathbb{A}_2^* are expanded like

$$\vec{r}_n = \sqrt{\frac{3}{2}}L \vec{\omega}_{(n_1, n_2)} \quad (6.5)$$

with

$$\vec{\omega}_{(n_1, n_2)} = n_1\vec{\omega}_1 + n_2\vec{\omega}_2 \quad (6.6)$$

and where L stands for the spacing parameter of the lattice and $n = (n_1, n_2)$ are arbitrary integers. Using (6.4), we also have

$$\vec{r}_n = \sqrt{\frac{3}{2}}L \frac{(2n_1 + n_2)}{3} \vec{\alpha}_1 + \sqrt{\frac{3}{2}}L \frac{(n_1 + 2n_2)}{3} \vec{\alpha}_2 \quad (6.7)$$

The architecture of the sites of the \mathbb{A}_2^* crystal is encoded into the intersection matrix

$$J_{ij}^{su_3} = \vec{\omega}_i \cdot \vec{\omega}_j \quad (6.8)$$

given by

$$J_{ij}^{su_3} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad (6.9)$$

with inverse

$$K_{ij}^{su_3} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad , \quad K_{ij}^{su_3} = \vec{\alpha}_i \cdot \vec{\alpha}_j \quad (6.10)$$

classifying closest neighbors

Each site \vec{r}_n in the base lattice \mathbb{A}_2^* has $(3 + 3)$ first nearest neighbors and 6 second nearest ones; they are as follows:

- *6 first nearests*

Up to a scaling factor L, the first nearest neighbors are given by

$$\begin{aligned} \vec{\lambda}_1 &= \vec{\omega}_1 \\ \vec{\lambda}_2 &= \vec{\omega}_2 - \vec{\omega}_1 \\ \vec{\lambda}_3 &= -\vec{\omega}_2 \end{aligned} \quad (6.11)$$

and

$$\begin{aligned} \vec{\zeta}_1 &= \vec{\omega}_2 \\ \vec{\zeta}_2 &= \vec{\omega}_1 - \vec{\omega}_2 \\ \vec{\zeta}_3 &= -\vec{\omega}_1 \end{aligned} \quad (6.12)$$

obeying the identities

$$\begin{aligned} \vec{\lambda}_1 + \vec{\lambda}_2 + \vec{\lambda}_3 &= \vec{0} \\ \vec{\zeta}_1 + \vec{\zeta}_2 + \vec{\zeta}_3 &= \vec{0} \end{aligned} \quad (6.13)$$

As an illustration on fig 2, choose as a node the green site at \vec{r}_n , its first nearest neighbors (6.11) are given by the red sites; and those of associated with (6.12) are given by the blue ones.

- *6 second nearest*

The 6 second nearest neighbors are given by

$$\begin{aligned}
\vec{\omega}_{(2,-1)} &= +2\vec{\omega}_1 - \vec{\omega}_2 \\
\vec{\omega}_{(1,1)} &= +\vec{\omega}_1 + \vec{\omega}_2 \\
\vec{\omega}_{(-1,2)} &= -\vec{\omega}_1 + 2\vec{\omega}_2 \\
\vec{\omega}_{(-2,1)} &= -2\vec{\omega}_1 + \vec{\omega}_2 \\
\vec{\omega}_{(-1,-1)} &= -\vec{\omega}_1 - \vec{\omega}_2 \\
\vec{\omega}_{(1,-2)} &= +\vec{\omega}_1 - 2\vec{\omega}_2
\end{aligned} \tag{6.14}$$

and are nothing but the six roots of the $SU(3)$ namely

$$\pm\vec{\alpha}_1 \quad , \quad \pm\vec{\alpha}_2 \quad , \quad \pm(\vec{\alpha}_1 + \vec{\alpha}_2) \tag{6.15}$$

As an illustration on fig 2, each green site \vec{r}_n has 6 nearest neighbors located at

$$\vec{r}_n \pm L\vec{\alpha}_1 \quad , \quad \vec{r}_n \pm L\vec{\alpha}_2 \quad , \quad \vec{r}_n \pm L(\vec{\alpha}_1 + \vec{\alpha}_2) \tag{6.16}$$

Observe also that a generic red site located at

$$\vec{r}_n + L\vec{\lambda}_i \quad , \quad i = 1, 2, 3 \tag{6.17}$$

has 6 first nearest neighbors given by the red ones located

$$\vec{r}_n + L\vec{\lambda}_i \pm L\vec{\alpha}_1 \quad , \quad \vec{r}_n + L\vec{\lambda}_i \pm L\vec{\alpha}_2 \quad , \quad \vec{r}_n + L\vec{\lambda}_i \pm L(\vec{\alpha}_1 + \vec{\alpha}_2) \tag{6.18}$$

This result may be also checked by computing the relative vector \vec{V}_{ij} between two nearest sites located at $\vec{r}_n + L\vec{\lambda}_i$ and $\vec{r}_n + L\vec{\lambda}_j$. We have

$$\begin{aligned}
\vec{V}_{ij} &= \left(\vec{r}_n + L\vec{\lambda}_i \right) - \left(\vec{r}_n + L\vec{\lambda}_j \right) \\
&= L \left(\vec{\lambda}_i - \vec{\lambda}_j \right)
\end{aligned} \tag{6.19}$$

which, by using eqs(6.12-6.14), the 6 vectors $\vec{\lambda}_i - \vec{\lambda}_j$ are precisely the 6 roots of $SU(3)$.

6.2 Building the lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$

The lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$ is a 3-dimensional crystal that may be thought of as given by a twisting of weight lattice \mathbb{A}_3^* of the Lie algebra of the symmetry

$$SO(6) \simeq SU(4) \quad (6.20)$$

To build this 3D crystal, we begin by describing the lattice \mathbb{A}_3^* ; after what we turn to construct $\mathcal{L}_{3D}^{su_3 \times u_1}$.

6.2.1 Construction of the lattice \mathbb{A}_3^*

The 3-dimensional lattice \mathbb{A}_3^* is the dual of the root lattice of $SU(4)$; it is generated by the 3 *fundamental weight* vectors of $SU(4)$

$$\vec{\Omega}_1, \quad \vec{\Omega}_2, \quad \vec{\Omega}_3 \quad (6.21)$$

Using the lattice spacing parameter L_{su_4} of the crystal \mathbb{A}_3^* , we can express the positions \vec{R}_n of sites in this lattice as follows

$$\vec{R}_n = L_{su_4} \sqrt{\frac{4}{3}} \vec{\Omega}_{(n_1, n_2, n_3)} \quad (6.22)$$

with

$$\vec{\Omega}_{(n_1, n_2, n_3)} = n_1 \vec{\Omega}_1 + n_2 \vec{\Omega}_2 + n_3 \vec{\Omega}_3 \quad (6.23)$$

where n_i are arbitrary integers.

The shape of \mathbb{A}_3^* is encoded in the intersection matrix

$$\mathcal{J}_{ij}^{su_4} = \vec{\Omega}_i \cdot \vec{\Omega}_j$$

given by

$$\mathcal{J}_{ij}^{su_4} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix} \quad (6.24)$$

The fundamental weight vectors $\vec{\Omega}_i$ are the dual of the 3 simple roots $\vec{a}_1, \vec{a}_2, \vec{a}_3$ of $SU(4)$ obeying

$$\vec{\Omega}_i \cdot \vec{a}_j = \delta_{ij} \quad (6.25)$$

The vectors $\vec{\Omega}_i$ are also the highest weight vectors of the complex $\mathbf{4}$, the adjoint conjugate $\bar{\mathbf{4}}$ and the real 6 dimensional representations of $SU(4)$.

$$\begin{array}{lll} \text{representation} & : & \mathbf{4} \quad \mathbf{6} \quad \bar{\mathbf{4}} \\ \text{highest weights} & : & \vec{\Omega}_1 \quad \vec{\Omega}_2 \quad \vec{\Omega}_3 \end{array} \quad (6.26)$$

The set of the weight vectors $\vec{\Lambda}_i$ defining the states of the complex 4- dimensional highest weight representations $\mathbf{4}$ and its conjugate $\bar{\mathbf{4}}$ are as follows

weight vectors of $\mathbf{4}$	weight vectors of $\bar{\mathbf{4}}$	
$\vec{\Lambda}_1 = \vec{\Omega}_1$	$\vec{\Lambda}'_1 = \vec{\Omega}_3$	(6.27)
$\vec{\Lambda}_2 = \vec{\Omega}_2 - \vec{\Omega}_1$	$\vec{\Lambda}'_2 = \vec{\Omega}_2 - \vec{\Omega}_3$	
$\vec{\Lambda}_3 = \vec{\Omega}_3 - \vec{\Omega}_2$	$\vec{\Lambda}'_3 = \vec{\Omega}_1 - \vec{\Omega}_2$	
$\vec{\Lambda}_4 = -\vec{\Omega}_3$	$\vec{\Lambda}'_4 = -\vec{\Omega}_1$	

These quartets obey the following traceless properties

$$\begin{aligned}\vec{\Lambda}_1 + \vec{\Lambda}_2 + \vec{\Lambda}_3 + \vec{\Lambda}_4 &= \vec{0} \\ \vec{\Lambda}'_1 + \vec{\Lambda}'_2 + \vec{\Lambda}'_3 + \vec{\Lambda}'_4 &= \vec{0}\end{aligned}\tag{6.28}$$

indicating that the sum of the relative positions of the first nearest neighbors should be equal to zero as schematized on fig 4. Similarly, the set of weight vectors $\vec{\Lambda}'_i$ of the real

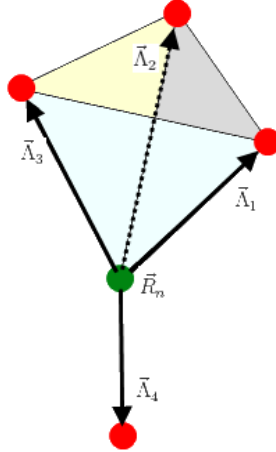


Figure 4: Each site \vec{R}_n in the lattice \mathbb{A}_3^* has $8 = 4 + 4$ first nearest neighbors associated with the weight vectors $\vec{\Lambda}_i$ and $\vec{\Lambda}'_i$. In the figure, 4 sites of the 8 ones are represented and are related to the Ω_i generators as in eqs(6.27).

6- dimensional representation of $SU(4) \simeq SO(6)$ are given by

$$\begin{aligned}
\vec{\Lambda}_1'' &= \vec{\Omega}_2 \\
\vec{\Lambda}_2'' &= \vec{\Omega}_1 + \vec{\Omega}_3 - \vec{\Omega}_2 \\
\vec{\Lambda}_3'' &= \vec{\Omega}_3 - \vec{\Omega}_1 \\
\vec{\Lambda}_4'' &= \vec{\Omega}_1 - \vec{\Omega}_3 \\
\vec{\Lambda}_5'' &= \vec{\Omega}_2 - \vec{\Omega}_1 - \vec{\Omega}_3 \\
\vec{\Lambda}_6'' &= -\vec{\Omega}_2
\end{aligned} \tag{6.29}$$

$$(6.30)$$

satisfying also a traceless property $\vec{\Lambda}_1'' + \dots + \vec{\Lambda}_6'' = 0$ and having an interpretation in terms of second nearest neighbors.

6.2.2 The lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$ as a twist of \mathbb{A}_3^*

Under the breaking of $SU(4)$ down to $U(1) \times SU(3)$, the representations $\mathbf{4}$ and $\bar{\mathbf{4}}$ break down to [44]

$$\begin{aligned}
\mathbf{4} &\rightarrow \mathbf{3}_{+q} \oplus \mathbf{1}_{-3q} \\
\bar{\mathbf{4}} &\rightarrow \bar{\mathbf{3}}_{-q} \oplus \bar{\mathbf{1}}_{+3q}
\end{aligned} \tag{6.31}$$

and the weight vectors (6.27) become

weight vectors of $\mathbf{3}_{+q} \oplus \mathbf{1}_{-3q}$	weight vectors of $\bar{\mathbf{3}}_{-q} \oplus \bar{\mathbf{1}}_{+3q}$	
$\vec{\Lambda}_1 = (\vec{\lambda}_1, +q)$	$\vec{\Lambda}'_0 = (\vec{0}, +3q)$	(6.32)
$\vec{\Lambda}_2 = (\vec{\lambda}_2, +q)$	$\vec{\Lambda}'_1 = (\vec{\zeta}_1, -q)$	
$\vec{\Lambda}_3 = (\vec{\lambda}_3, +q)$	$\vec{\Lambda}'_2 = (\vec{\zeta}_2, -q)$	
$\vec{\Lambda}_0 = (0, 0, -3q)$	$\vec{\Lambda}'_3 = (\vec{\zeta}_3, -q)$	

where where $\vec{\lambda}_i, \vec{\zeta}_i$ are as in eqs(6.11).

Similarly, we have for the 6-dimensional representation the following decomposition

$$\mathbf{6} \rightarrow \mathbf{3}_{-2q} \oplus \bar{\mathbf{3}}_{+2q} \tag{6.33}$$

with weight vectors $\vec{\Lambda}_i''$ as follows

$$\begin{aligned}
\vec{\Lambda}_1'' &= \begin{pmatrix} \vec{\zeta}_1, +2q \end{pmatrix} \\
\vec{\Lambda}_2'' &= \begin{pmatrix} \vec{\zeta}_2, +2q \end{pmatrix} \\
\vec{\Lambda}_3'' &= \begin{pmatrix} \vec{\zeta}_3, +2q \end{pmatrix} \\
\vec{\Lambda}_4'' &= \begin{pmatrix} \vec{\lambda}_1, -2q \end{pmatrix} \\
\vec{\Lambda}_5'' &= \begin{pmatrix} \vec{\lambda}_2, -2q \end{pmatrix} \\
\vec{\Lambda}_6'' &= \begin{pmatrix} \vec{\lambda}_3, -2q \end{pmatrix}
\end{aligned} \tag{6.34}$$

From these decompositions, we deduce the expressions of the fundamental weight vectors of $U(1) \times SU(3)$ that we denote as $\vec{\Gamma}_1, \vec{\Gamma}_2, \vec{\Gamma}_3$. These vectors are given by

$$\begin{aligned}
\vec{\Gamma}_1 &= (\vec{\omega}_1, q) \\
\vec{\Gamma}_2 &= (\vec{\omega}_2, 2q) \\
\vec{\Gamma}_0 &= (0, 0, 3q)
\end{aligned} \tag{6.35}$$

with unit charge q to be fixed later. Computing the intersection matrix of these weights

$$\mathcal{J}_{ij}^{su_3 \times u_1} = \vec{\Gamma}_i \cdot \vec{\Gamma}_j \tag{6.36}$$

we find

$$\mathcal{J}_{ij}^{su_3 \times u_1} = \begin{pmatrix} \frac{2}{3} + q^2 & \frac{1}{3} + 2q^2 & 3q^2 \\ \frac{1}{3} + 2q^2 & \frac{2}{3} + 4q^2 & 6q^2 \\ 3q^2 & 6q^2 & 9q^2 \end{pmatrix} \tag{6.37}$$

with

$$\det \mathcal{J}_{ij}^{su_3 \times u_1} = 3q^2 \tag{6.38}$$

and inverse given by

$$\mathcal{K}_{ij}^{su_3 \times u_1} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & \frac{6q^2+1}{9q^2} \end{pmatrix} \tag{6.39}$$

By requiring the following condition on the self intersection of $\vec{\Gamma}_3$

$$\frac{6q^2+1}{9q^2} = 1 \tag{6.40}$$

it results

$$q^2 = \frac{1}{3} \tag{6.41}$$

Putting this value back into the above intersection matrices, we end with

$$\mathcal{J}_{ij}^{su_3 \times u_1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \quad (6.42)$$

and

$$\mathcal{K}_{ij}^{su_3 \times u_1} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad (6.43)$$

From these intersection matrices, we determine the following relations

$$\begin{aligned} \vec{a}_1 &= 2\vec{\Gamma}_1 - \vec{\Gamma}_2 \\ \vec{a}_2 &= 2\vec{\Gamma}_2 - \vec{\Gamma}_1 - \vec{\Gamma}_0 \\ \vec{a}_0 &= \vec{\Gamma}_0 - \vec{\Gamma}_2 \end{aligned} \quad (6.44)$$

and

$$\begin{aligned} \vec{\Gamma}_1 &= \vec{a}_1 + \vec{a}_2 + \vec{a}_0 \\ \vec{\Gamma}_2 &= \vec{a}_1 + 2\vec{a}_2 + 2\vec{a}_0 \\ \vec{\Gamma}_0 &= \vec{a}_1 + 2\vec{a}_2 + 3\vec{a}_0 \end{aligned} \quad (6.45)$$

Sites \vec{R}_n in $\mathcal{L}_{3D}^{su_3 \times u_1}$ are therefore expanded like

$$\vec{R}_n = L \vec{\Gamma}_n, \quad L = L_{su_4} \sqrt{\frac{4}{3}} \quad (6.46)$$

with

$$\vec{\Gamma}_n = n_1 \vec{\Gamma}_1 + n_2 \vec{\Gamma}_2 + n_0 \vec{\Gamma}_0 \quad (6.47)$$

and

$$\vec{\Gamma}_1 = \begin{pmatrix} \vec{\omega}_1 \\ q \end{pmatrix}, \quad \vec{\Gamma}_2 = \begin{pmatrix} \vec{\omega}_2 \\ 2q \end{pmatrix}, \quad \vec{\Gamma}_0 = \begin{pmatrix} \vec{0} \\ 3q \end{pmatrix} \quad (6.48)$$

obeying the duality relation

$$\vec{\Gamma}_i \cdot \vec{a}_j = \delta_{ij} \quad (6.49)$$

We also have

$$\vec{a}_1 = \begin{pmatrix} 2\vec{\omega}_1 - \vec{\omega}_2 \\ 0 \end{pmatrix}, \quad \vec{a}_2 = \begin{pmatrix} 2\vec{\omega}_2 - \vec{\omega}_1 \\ 0 \end{pmatrix}, \quad \vec{a}_0 = \begin{pmatrix} -\vec{\omega}_2 \\ q \end{pmatrix} \quad (6.50)$$

6.3 Lattice interpretation of BRST symmetry

Here we use the weight vectors of the $SU(3) \times U(1)$ representations carried by the fermionic and the bosonic fields of the spectrum of twisted supersymmetric YM theory on $\mathcal{L}_{3D}^{su_3 \times u_1}$ to give a crystal interpretation of the BRST symmetry. To that purpose, we first derive the closed nearest neighbors in $\mathcal{L}_{3D}^{su_3 \times u_1}$; then, we give the lattice realization of BRST symmetry.

6.3.1 Closest neighbors in $\mathcal{L}_{3D}^{su_3 \times u_1}$

Given a generic site \vec{R}_n in $\mathcal{L}_{3D}^{su_3 \times u_1}$, its closest neighbors are as follows:

a) *first nearest neighbors and the links $\psi_n^{I(+q)}$:*

Each site in $\mathcal{L}_{3D}^{su_3 \times u_1}$ has 6 first nearest neighbors that split into 3 + 3 ones respectively located at

$$\vec{R}_n + L\vec{\Lambda}_I = \vec{R}_n + \vec{V}_I^{(+q)} \quad (6.51)$$

and

$$\vec{R}_n + L\vec{\Lambda}'_I = \vec{R}_n - \vec{V}_I^{(+q)} \quad (6.52)$$

with

$$\begin{aligned} \vec{V}_1^{(+q)} &= L\vec{\Gamma}_1 \\ \vec{V}_2^{(+q)} &= L\left(\vec{\Gamma}_2 - \vec{\Gamma}_1\right) \\ \vec{V}_3^{(+q)} &= L\left(\vec{\Gamma}_0 - \vec{\Gamma}_2\right) \end{aligned} \quad (6.53)$$

satisfying

$$\vec{V}_1^{(+q)} + \vec{V}_2^{(+q)} + \vec{V}_3^{(+q)} = L\vec{\Gamma}_0 \quad (6.54)$$

Using (6.48), these relations read also like

$$\begin{aligned} \vec{V}_1^{(+q)} &= L\left(\vec{\lambda}_1, q\right) \\ \vec{V}_2^{(+q)} &= L\left(\vec{\lambda}_2, q\right) \\ \vec{V}_3^{(+q)} &= L\left(\vec{\lambda}_3, q\right) \end{aligned} \quad (6.55)$$

with

$$\vec{V}_1^{(+q)} + \vec{V}_2^{(+q)} + \vec{V}_3^{(+q)} = (0, 0, 3Lq) \quad (6.56)$$

and length

$$\begin{aligned} \left(\vec{V}_I^{(+q)}\right)^2 &= \left(-\vec{V}_I^{(+q)}\right)^2 \\ &= \left(\frac{2}{3} + q^2\right)L^2 = L^2 \end{aligned} \quad (6.57)$$

since $3q^2 = 1$.

Therefore the oriented links

$$\overrightarrow{P_{R_n} P_{R_n + V_I^{(+q)}}} \sim V_I^{(+q)} \quad (6.58)$$

are associated with the fermionic lattice variables

$$\psi_n^{I(+q)} = \psi^{I(+q)}(R_n) \quad (6.59)$$

b) second nearest neighbors and gauge field variables

Each site R_n in $\mathcal{L}_{3D}^{su_3 \times u_1}$ has 6 second nearest neighbors that also split into 3 + 3 ones respectively located at

$$\vec{R}_n + \vec{V}_I^{(-2q)} \quad (6.60)$$

and at

$$\begin{aligned} \vec{R}_n + \vec{V}_I^{(+2q)} &= \vec{R}_n - \vec{V}_I^{(-2q)} \\ &\sim -\vec{V}_I^{(-2q)} \end{aligned} \quad (6.61)$$

Upon the normalization $L = 1$, we have

$$\begin{aligned} \vec{V}_1^{(+2q)} &= \vec{\Gamma}_2 \\ \vec{V}_2^{(+2q)} &= \vec{\Gamma}_1 + \vec{\Gamma}_0 - \vec{\Gamma}_2 \\ \vec{V}_3^{(+2q)} &= \vec{\Gamma}_0 - \vec{\Gamma}_1 \\ \vec{V}_1^{(-2q)} &= \vec{\Gamma}_1 - \vec{\Gamma}_0 \\ \vec{V}_2^{(-2q)} &= \vec{\Gamma}_2 - \vec{\Gamma}_1 - \vec{\Gamma}_0 \\ \vec{V}_3^{(-2q)} &= -\vec{\Gamma}_2 \end{aligned} \quad (6.62)$$

or equivalently

$$\begin{aligned} \vec{V}_1^{(+2q)} &= \left(-\vec{\lambda}_1, 2q \right) \\ \vec{V}_2^{(+2q)} &= \left(-\vec{\lambda}_2, 2q \right) \\ \vec{V}_3^{(+2q)} &= \left(-\vec{\lambda}_3, 2q \right) \\ \vec{V}_1^{(-2q)} &= \left(\vec{\lambda}_1, -2q \right) \\ \vec{V}_2^{(-2q)} &= \left(\vec{\lambda}_2, -2q \right) \\ \vec{V}_3^{(-2q)} &= \left(\vec{\lambda}_3, -2q \right) \end{aligned} \quad (6.63)$$

with length

$$\begin{aligned} \left(\vec{V}_a^{(+2q)} \right)^2 &= \left(\frac{2}{3} + 4q^2 \right) L^2 = 2L^2 \\ \left(\vec{V}_a^{(-2q)} \right)^2 &= \left(\frac{2}{3} + 4q^2 \right) L^2 = 2L^2 \end{aligned} \quad (6.64)$$

The 3 oriented links

$$\overrightarrow{P_{R_n} P_{R_n + V_I^{(+2q)}}} \sim -\vec{V}_I^{(-2q)} \quad (6.65)$$

are associated with the gauge field variables

$$U_{n,I}^{(+2q)} = U_I^{(+2q)}(R_n)$$

while the opposite ones

$$\begin{aligned} \overrightarrow{P_{R_n} P_{R_n - V_I^{(+2q)}}} &\equiv P_{R_n + V_I^{(-2q)}} P_{R_n} \\ &\sim V_I^{(-2q)} \end{aligned} \quad (6.66)$$

with the complex conjugate fields

$$U_n^{I(-2q)} = U^{I(-2q)}(R_n) \quad (6.67)$$

c) third nearest neighbors

The site \vec{R}_n of $\mathcal{L}_{3D}^{su_3 \times u_1}$ has 2 third nearest neighbors that also split into 1 + 1 respectively located at

$$\begin{aligned} \vec{R}_n + L\vec{\Gamma}_0 &= \vec{R}_n + \vec{V}^{(+3q)} \\ \vec{R}_n - L\vec{\Gamma}_0 &= \vec{R}_n + \vec{V}^{(-3q)} \end{aligned} \quad (6.68)$$

with

$$\vec{\Gamma}_0 = \begin{pmatrix} \vec{0} \\ 3q \end{pmatrix} \quad (6.69)$$

and

$$\vec{\Gamma}_0 \cdot \vec{\Gamma}_0 = 3 \quad (6.70)$$

The oriented links

$$\overrightarrow{P_{R_n} P_{R_n + V^{(-3q)}}} \sim V^{(-3q)} \quad (6.71)$$

are associated with the fermionic singlets $\psi_n^{(-3q)}$ and the Grassman variable

$$\psi_n^{(-3q)} = \psi^{(-3q)}(R_n) \quad , \quad \theta^{(-3q)} \quad (6.72)$$

while the opposite ones

$$\begin{aligned} \overrightarrow{P_{R_n} P_{R_n - V^{(-3q)}}} &\equiv P_{R_n + V^{(+3q)}} P_{R_n} \\ &\sim -V^{(-3q)} \end{aligned} \quad (6.73)$$

with objects carrying +3 unit charges of U(1) like the discrete analogue of $\nabla_a^{(+2q)} \psi^{a(+)}$.

6.3.2 BRST symmetry on lattice

Being a scalar object under $SU(3)$ but having non trivial charges of $U(1)$, the scalar supersymmetric $Q^{(+3q)}$ may be interpreted as a link operator living on the direction

$$\vec{\Gamma} = (0, 0, 3q) \quad (6.74)$$

Therefore supersymmetric transformations generated by $Q^{(+3q)}$ can be interpreted as particular shifts on the lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$; more precisely shifts in the $\mathcal{L}_{1D}^{u_1}$ sublattice of the fibration

$$\begin{array}{ccc} \mathcal{L}_{1D}^{u_1} & \rightarrow & \mathcal{L}_{3D}^{su_3 \times u_1} \\ & & \downarrow \\ & & \mathbb{A}_2^* \end{array}$$

For example, the supersymmetric transformation

$$Q^{(+3q)} U_n^{I(-2q)} = \psi_n^{I(+q)} \quad (6.75)$$

corresponds to shifting $\vec{V}_I^{(-2q)}$ as follows

$$\vec{V}_I^{(-2q)} + \vec{V}_0^{(+3q)} = \vec{V}_I^{(+q)} \quad (6.76)$$

In general, the operator $Q^{(+3q)}$ acts on the various lattice variables like

$$\begin{aligned} Q^{(+3q)} U_n^{I(-2q)} &= \psi_n^{I(+q)} \\ Q^{(+3q)} \psi_n^{I(+q)} &= 0 \\ \\ Q^{(+3q)} U_{I,n}^{(+2q)} &= 0 \\ Q^{(+3q)} W_{IJ,n}^{(+4q)} &= 0 \\ \\ Q^{(+3q)} \psi_n^{(-3q)} &= F_n^{(0)} \\ Q^{(+3q)} F_n^{(0)} &= 0 \\ \\ Q^{(+3q)} W_n^{IJ(-4q)} &= \Upsilon_n^{IJ(-q)} \\ Q^{(+3q)} \Upsilon_n^{IJ(-q)} &= 0 \\ Q^{(+3q)} W_n^{(0)} &= \Upsilon_n^{(+3q)} \\ Q^{(+3q)} \Upsilon_n^{(+3q)} &= 0 \end{aligned} \quad (6.77)$$

with

$$\begin{aligned}
\Upsilon_n^{IJ(-q)} &= U^{I(-2q)}(R_n) \psi^{J(+q)}(R_n + \vec{V}_I^{(-2q)}) - \\
&\quad \psi^{J(+q)}(R_n) U^{I(-2q)}(R_n + \vec{V}_J^{(-2q)}) \\
\Upsilon_n^{(0)} &= \psi^{I(+q)}(R_n) U_I^{(+2q)}(R_n) - \\
&\quad U_I^{(+2q)}(R_n - \vec{V}_I^{(-2q)}) \psi^{I(+q)}(R_n - \vec{V}_I^{(-2q)})
\end{aligned} \tag{6.78}$$

7 Action of 3D $\mathcal{N} = 4$ on Lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$

First we give the expression of the fields on the lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$; then we build the gauge invariant action on $\mathcal{L}_{3D}^{su_3 \times u_1}$.

7.1 Fields on $\mathcal{L}_{3D}^{su_3 \times u_1}$: a dictionary

Under discretization of the the complex 3D continuum into the crystal $\mathcal{L}_{3D}^{su_3 \times u_1}$, the analogue of coordinate variables and fields are given by the following dictionary

- *Coordinates variables*

$$\begin{array}{ll}
\text{continuum} & : \quad \text{crystal } \mathcal{L}_{3D}^{su_3 \times u_1} \\
(z^{a(-2q)}, \bar{z}_a^{(+2q)}) & \vec{R}_n \\
z^{a(-2q)} + dz^{a(-2q)} & \vec{R}_n + \vec{V}_I^{(-2q)} \\
\bar{z}_a^{(+2q)} + d\bar{z}_a^{(+2q)} & \vec{R}_n + \vec{V}_I^{(+2q)} = \vec{R}_n - \vec{V}_I^{(-2q)}
\end{array} \tag{7.1}$$

with, by setting $L = 1$,

$$\begin{aligned}
\vec{V}_I^{(+2q)} &= \left(-\vec{\lambda}_a, +2q \right) \\
\vec{V}_I^{(-2q)} &= \left(+\vec{\lambda}_a, -2q \right)
\end{aligned} \tag{7.2}$$

- *Bosonic fields*

$$\begin{array}{ll}
\text{continuum} & : \quad \text{crystal } \mathcal{L}_{3D}^{su_3 \times u_1} \\
\mathcal{G}^{a(-2q)}(z, \bar{z}) & U_n^{I(-2q)} \\
\mathcal{G}_a^{(+2q)}(z, \bar{z}) & U_{n,I}^{(+2q)} \\
\mathcal{F}^{ab(-4q)}(z, \bar{z}) & W_n^{IJ(-4q)} \\
\bar{\mathcal{F}}_{ab}^{(+4q)}(z, \bar{z}) & W_{n,IJ}^{(+4q)} \\
\mathcal{F}^{(0)}(z, \bar{z}) & W_n^{(0)}
\end{array} \tag{7.3}$$

with

$$\begin{aligned}
U_n^{I(-2q)} &= U^{I(-2q)}(R_n) \\
U_{n,I}^{(+2q)} &= U_I^{(+2q)}(R_n) \\
W_n^{IJ(-4q)} &= U^{I(-2q)}(R_n) U^{J(-2q)}(R_n + V_I^{(-2q)}) - \\
&\quad U^{J(-2q)}(R_n) U^{I(-2q)}(R_n + V_J^{(-2q)}) \\
W_{n,IJ}^{(+4q)} &= U_J^{(+2q)}(R_n + V_I^{(-2q)}) U_I^{(+2q)}(R_n) - \\
&\quad U_I^{(+2q)}(R_n + V_J^{(-2q)}) U_J^{(+2q)}(R_n)
\end{aligned} \tag{7.4}$$

$$\begin{aligned}
W_n^{(0)} &= U^{I(-2q)}(R_n) U_I^{(+2q)}(R_n) - \\
&\quad U_I^{(+2q)}(R_n - V_I^{(-2q)}) U^{I(-2q)}(R_n - V_I^{(-2q)})
\end{aligned} \tag{7.5}$$

- *Fermionic fields*

$$\begin{array}{ll}
\text{continuum} & : \quad \text{crystal } \mathcal{L}_{3D}^{su_3 \times u_1} \\
\psi^{(-3q)}(z, \bar{z}) & \psi_n^{(-3q)} \\
\psi^{I(+q)}(z, \bar{z}) & \psi_n^{I(+q)} \\
\nabla^{a(-2q)} \psi^{b(+)} & \psi_n^{IJ(-q)} \\
\bar{\nabla}_a^{(+2q)} \psi^{a(-)} & \psi_{n,I}^{I(+)}
\end{array} \tag{7.6}$$

with

$$\begin{aligned}
\psi_n^{(-3q)} &= \psi^{(-3q)}(R_n) \\
\psi_n^{I(+q)} &= \psi^{I(+q)}(R_n) \\
\psi_n^{IJ(-q)} &= U^{I(-2q)}(R_n + V_J^{(-q)}) \psi^{J(+q)}(R_n + V_I^{(-2q)}) - \\
&\quad \psi^{J(+q)}(R_n) U^{I(-2q)}(R_n) \\
\psi_{n,I}^{I(+3q)} &= \psi^{I(+q)}(R_n + V_I^{(-2q)}) U_I^{(+2q)}(R_n) - \\
&\quad U_I^{(+2q)}(R_n - V_I^{(-q)}) \psi^{I(+q)}(R_n)
\end{aligned} \tag{7.7}$$

- *Gauge symmetry*

lattice variable	→	gauge transform	
$U_n^{I(-2q)}$		$G(R_n) U^{I(-2q)}(R_n) G^\dagger(R_n + V_I^{(-2q)})$	
$U_{n,I}^{(+2q)}$		$G(R_n + V_I^{(-2q)}) U_I^{(+2q)}(R_n) G^\dagger(R_n)$	
$W_n^{IJ(-4q)}$		$G(R_n) W_n^{IJ(-4q)} G^\dagger(R_n + V_I^{(-2q)} + V_J^{(-2q)})$	
$W_{n,IJ}^{(+4q)}$		$G(R_n + V_I^{(-2q)} + V_J^{(-2q)}) W_n^{IJ(+4q)} G^\dagger(R_n)$	(7.8)
$\psi_n^{(-3q)}$		$G(R_n) \psi^{(-3q)}(R_n) G^\dagger(R_n + V_0^{(-3q)})$	
$\psi_n^{I(+q)}$		$G(R_n + V_I^{(-q)}) \psi^{I(+q)}(R_n) G^\dagger(R_n)$	
$\psi_n^{IJ(-q)}$		$G(R_n + V_J^{(-q)}) \psi^{IJ(-q)}(R_n) G^\dagger(R_n + V_I^{(-2q)})$	
$\psi_{n,I}^{I(+3q)}$		$G(R_n + V_I^{(-q)} + V_I^{(-2q)}) \psi_I^{I(+3q)}(R_n) G^\dagger(R_n)$	

7.2 Useful identities

Here we collect some relations useful for checking gauge invariance of the lattice field action $\mathcal{L}_{3D}^{su_3 \times u_1}$

$$\begin{aligned}
\begin{aligned}
\vec{V}_1^{(+q)} &= \begin{pmatrix} \vec{\lambda}_1, q \end{pmatrix} \\
\vec{V}_2^{(+q)} &= \begin{pmatrix} \vec{\lambda}_2, q \end{pmatrix} \\
\vec{V}_3^{(+q)} &= \begin{pmatrix} \vec{\lambda}_3, q \end{pmatrix}
\end{aligned}
, \quad
\vec{V}_I^{(-q)} = -\vec{V}_I^{(+q)} \\
\\
\begin{aligned}
\vec{V}_1^{(+2q)} &= \begin{pmatrix} -\vec{\lambda}_1, 2q \end{pmatrix} \\
\vec{V}_2^{(+2q)} &= \begin{pmatrix} -\vec{\lambda}_2, 2q \end{pmatrix} \\
\vec{V}_3^{(+2q)} &= \begin{pmatrix} -\vec{\lambda}_3, 2q \end{pmatrix}
\end{aligned}
, \quad
\vec{V}_I^{(-2q)} = -\vec{V}_I^{(+2q)} \\
\\
\vec{V}_0^{(+3q)} &= \begin{pmatrix} \vec{0}, 3q \end{pmatrix}
, \quad
\vec{V}_0^{(-3q)} = -\vec{V}_0^{(+3q)}
\end{aligned} \tag{7.9}$$

with

$$\vec{\lambda}_3 = -\vec{\lambda}_1 - \vec{\lambda}_2 \quad (7.10)$$

From these relations, we learn a set of identities; in particular

$$\begin{aligned} V_1^{(+q)} + V_2^{(+q)} + V_3^{(-2q)} &= \vec{0} \\ V_1^{(+q)} + V_2^{(-2q)} + V_3^{(+q)} &= \vec{0} \\ V_1^{(-2q)} + V_2^{(+q)} + V_3^{(+q)} &= \vec{0} \end{aligned} \quad (7.11)$$

and

$$\begin{aligned} \vec{V}_1^{(+q)} + \vec{V}_2^{(+q)} + \vec{V}_3^{(+q)} &= \vec{V}_0^{(+3q)} \\ \vec{V}_1^{(-q)} + \vec{V}_2^{(-q)} + \vec{V}_3^{(-q)} &= \vec{V}_0^{(-3q)} \\ \vec{V}_I^{(+q)} + \vec{V}_I^{(+2q)} &= \vec{V}_0^{(+3q)} \\ \vec{V}_I^{(-q)} + \vec{V}_I^{(-2q)} &= \vec{V}_0^{(-3q)} \end{aligned} \quad (7.12)$$

These identities are important for establishing gauge symmetry of monomials like

$$\begin{aligned} (i) \quad &: \quad Tr \left[\psi^{(-3q)} (R_n) \psi_I^{I(+3q)} (R_n) \right] \\ (ii) \quad &: \quad \varepsilon_{IJK} Tr \left[\psi^{I(+q)} (R_n - V_I^{(-q)}) \psi^{JK(-q)} (R_n) \right] \end{aligned} \quad (7.13)$$

- *checking gauge invariance of the term (i)*

Under a gauge transformation $\mathbf{G}(R_n)$, the lattice variables $\psi_n^{(-3q)}$ and the divergence $\psi_{n,I}^{I(+3q)}$ transform as

$$\begin{aligned} \psi_n^{(-3q)} &\rightarrow G(R_n) \psi^{(-3q)}(R_n) G^\dagger(R_n + V_0^{(-3q)}) \\ \psi_{n,I}^{I(+3q)} &\rightarrow G(R_n + V_I^{(-q)} + V_I^{(-2q)}) \psi_I^{I(+3q)}(R_n) G^\dagger(R_n) \end{aligned} \quad (7.14)$$

Using the identity

$$V_I^{(-q)} + V_I^{(-2q)} = V_0^{(-3q)} \quad (7.15)$$

the second term of (7.14) reads also like

$$\psi_{n,I}^{I(+3q)} \rightarrow G(R_n + V_0^{(-3q)}) \psi_I^{I(+3q)}(R_n) G^\dagger(R_n) \quad (7.16)$$

Therefore the gauge transformation of the term $\psi^{(-3q)}(R_n) \psi_I^{I(+3q)}(R_n)$ is given by

$$Tr \left[G(R_n) \psi^{(-3q)}(R_n) G^\dagger(R_n + V_0^{(-3q)}) G(R_n + V_0^{(-3q)}) \psi_I^{I(+3q)}(R_n) G^\dagger(R_n) \right]$$

and leads then to

$$Tr \left[G(R_n) \psi^{(-3q)}(R_n) \psi_I^{I(+3q)}(R_n) G^\dagger(R_n) \right] \quad (7.17)$$

which, due to the cyclic property of the trace, reduce further to

$$Tr \left[\psi^{(-3q)}(R_n) \psi_I^{I(+3q)}(R_n) \right] \quad (7.18)$$

- *checking gauge invariance of the term (ii)*

Starting from the gauge transformation of the lattice variables $\psi_n^{I(+q)}$ and $\psi_n^{IJ(-q)}$ namely

$$\begin{aligned} \psi_n^{I(+q)} &\rightarrow G(R_n + V_I^{(-q)}) \psi^{I(+q)}(R_n) G^\dagger(R_n) \\ \psi_n^{JK(-q)} &\rightarrow G(R_n + V_K^{(-q)}) \psi^{JK(-q)}(R_n) G^\dagger(R_n + V_J^{(-2q)}) \end{aligned} \quad (7.19)$$

the putting back into

$$\varepsilon_{IJK} Tr \left[\psi^{I(+q)}(R_n + V_K^{(-q)}) \psi^{JK(-q)}(R_n) \right] \quad (7.20)$$

we get

$$\begin{aligned} &G(R_n + V_K^{(-q)} + V_I^{(-q)}) \psi^{I(+q)}(R_n + V_K^{(-q)}) G^\dagger(R_n + V_K^{(-q)}) \times \\ &G(R_n + V_K^{(-q)}) \psi^{JK(-q)}(R_n) G^\dagger(R_n + V_J^{(-2q)}) \end{aligned} \quad (7.21)$$

Invariance of the trace of this quantity requires

$$G^{\dagger\dagger}(R_n + V_J^{(-2q)}) G(R_n + V_K^{(-q)} + V_I^{(-q)}) = I_{id} \quad (7.22)$$

requiring in turn the constraint relation

$$V_K^{(-q)} + V_I^{(-q)} = V_J^{(-2q)} \quad , \quad I \neq J \neq K \quad (7.23)$$

which is identically satisfied due to eqs(7.9).

7.3 The action on $\mathcal{L}_{3D}^{su_3 \times u_1}$

Starting from the twisted Lagrangian density \mathbf{L}_{twist} (4.17) namely

$$\begin{aligned} \mathbf{L}_{twist} &= Tr \left[\mathcal{F}_{ab}^{(+4q)} \mathcal{F}^{ab(-4q)} \right] \\ &\quad - \frac{(\alpha_2)^2}{4\alpha_1\alpha_4} Tr \left[\mathcal{F}^{(0)} \mathcal{F}^{(0)} \right] \\ &\quad + \left(\frac{\alpha_2}{\alpha_4} + 2 \right) Tr \left[\psi^{a(+q)} \nabla_a^{(+2q)} \psi^{(-3q)} \right] \\ &\quad + 2 \frac{\alpha_3}{\alpha_4} Tr \left[\varepsilon_{abc} \psi^{a(+q)} \nabla^b(-2q) \psi^{c(+q)} \right] \end{aligned} \quad (7.24)$$

and using the dictionary of subsection 6.1 between fields in continuum and lattice variables, we can write down the gauge invariant action on $\mathcal{L}_{3D}^{su_3 \times u_1}$ following from the above one. We find:

$$\begin{aligned} \mathcal{S}_{latt} &= \sum_{\mathcal{L}_{3D}^{su_3 \times u_1}} Tr \left(W_n^{IJ(-4q)} W_{n,IJ}^{(+4q)} \right) - \frac{(\alpha_2)^2}{4\alpha_1\alpha_4} \sum_{\mathcal{L}_{3D}^{su_3 \times u_1}} Tr \left(W_n^{(0)} W_n^{\dagger(0)} \right) \\ &\quad \left(2 + \frac{\alpha_2}{\alpha_4} \right) \sum_{\mathcal{L}_{3D}^{su_3 \times u_1}} Tr \left[\psi^{(-3q)}(R_n) \psi^{I(+q)}(R_n + V_I^{(-2q)}) U_I^{(+2q)}(R_n) \right] + \\ &\quad \left(2 + \frac{\alpha_2}{\alpha_4} \right) \sum_{\mathcal{L}_{3D}^{su_3 \times u_1}} Tr \left[\psi^{(-3q)}(R_n) U_I^{(+2q)}(R_n - V_I^{(-q)}) \psi^{I(+q)}(R_n) \right] \\ &\quad + 2 \frac{\alpha_3}{\alpha_4} \sum_{\mathcal{L}_{3D}^{su_3 \times u_1}} \varepsilon_{IJK} Tr \left[\psi^{I(+q)}(R_n + V_K^{(-q)}) \psi^{JK(-q)}(R_n) \right] \end{aligned} \quad (7.25)$$

with $W_n^{IJ(-4q)}$, $W_{n,IJ}^{(+4q)}$, $W_n^{(0)}$ and $\psi^{JK(-q)}(R_n)$ as in eqs(7.7) and where α_i 's are normalization numbers.

8 Conclusion and comments

In this paper, we studied twisted $3D \mathcal{N} = 4$ supersymmetric YM on a particular 3-dimensional lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$ having an $SU(3) \times U(1)$ symmetry and realized by the fibration

$$\mathcal{L}_{2D}^{su_3} \times \mathcal{L}_{1D}^{u_1}$$

with the 2-dimensional base $\mathcal{L}_{2D}^{su_3} = \mathbb{A}_2^*$, the weight lattice of $SU(3)$, and fiber $\mathcal{L}_{1D}^{u_1} \simeq q\mathbb{Z}$. This fibration is encoded by the intersection matrix

$$\mathcal{J}_{ij}^{su_3 \times u_1} = \begin{pmatrix} \frac{2}{3} + q^2 & \frac{1}{3} + 2q^2 & 3q^2 \\ \frac{1}{3} + 2q^2 & \frac{2}{3} + 4q^2 & 6q^2 \\ 3q^2 & 6q^2 & 9q^2 \end{pmatrix}, \quad \det \mathcal{J}_{ij}^{su_3 \times u_1} = 3q^2$$

with q a unit charge of $U(1)$. The $SU(3) \times U(1)$ complex symmetry appearing here is one of the breaking modes of the $SO_E(6)$ symmetry of the chiral $6D \mathcal{N} = 1$ supersymmetric YM on \mathbb{R}^6 ; the usual breaking mode used in the twisting is given by the real

$$SO_E(3) \times SO_R(3)$$

symmetry with $SO_E(3)$ the isotropy group of \mathbb{R}^3 and $SO_R(3)$ the R-symmetry. The group $SU(3) \times U(1)$ may be therefore viewed as a complexification of the diagonal symmetry of $SO_E(3) \times SO_R(3)$.

To that purpose, we first reviewed general aspects of $SO(t, s)$ spinors in diverse dimensions; then we built the twisted $3D \mathcal{N} = 4$ supersymmetric algebra (3.3) generated, in addition to the bosonic, by 4 complex fermionic generators

$$Q^{(+3q)}, \quad Q_a^{(-q)}$$

transforming respectively as a complex $SU(3)$ singlet and a complex $SU(3)$ triplet carrying moreover non trivial charges under $U(1)$, the number q is a non zero unit charge of $U(1)$; but its singular limit

$$q = 0$$

has an interpretation on lattice; it corresponds to the projection of $\mathcal{L}_{3D}^{su_3 \times u_1}$ down to the base sublattice $\mathcal{L}_{2D}^{su_3} = \mathbb{A}_2^*$.

Then extending ideas from covariant gauge formalism of supersymmetric YM theories

and using the gauge covariant superfields (3.20), we studied the superspace formulation of the twisted gauge theory exhibiting manifestly invariance under $Q^{(+3q)}$. This supercharge may be also interpreted as a particular BRST operator and the corresponding supersymmetric transformation as BRST transformations. The derivation of the set of gauge covariant superfields (3.20) is a key step in our construction since only 1 of the 4 complex (8 real) supersymmetric charges are off shell; this set is explicitly derived in the appendix, eqs(9.7-9.8).

After that, we studied the lattice version of twisted $3D \mathcal{N} = 4$ supersymmetric YM living on \mathcal{L}_{3D} given by the fibration

$$\begin{array}{ccc} \mathcal{L}_{1D}^{u(1)} & \rightarrow & \mathcal{L}_{3D}^{su_3 \times u_1} \\ & & \downarrow \\ & & \mathcal{L}_{2D}^{su_3} = \mathbb{A}_2^* \end{array} \quad (8.1)$$

To achieve the lattice construction, we performed the 3 following steps:

- (a) developed a method of engineering the crystal $\mathcal{L}_{3D}^{su_3 \times u_1}$ with a manifestly $SU(3) \times U(1)$ symmetry. This lattice is given by the fibration (8.1); the shape of the base sublattice \mathbb{A}_2^* , corresponds to the projection $q = 0$, and is completely given by the inverse of the Cartan matrix of $SU(3)$. The 3D lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$ is a twist of the weight lattice \mathbb{A}_3^* of $SU(4) \simeq SO(6)$

$$\mathcal{L}_{3D}^{su_3 \times u_1} \sim \text{twist of } \mathbb{A}_3^*$$

- (b) worked out the dictionary eqs(7.1-7.8) between objects \mathcal{O}_{cont} living in continuum and their analogue $\mathcal{O}_{lattice}$ on the lattice $\mathcal{L}_{3D}^{su_3 \times u_1}$. The objects include the twisted fields, the coordinates and the supersymmetric generators.
- (c) built the lattice action $\mathcal{S}_{lattice}$ that is invariant under:
 - (i) the $U(N)$ gauge symmetry,
 - (ii) the complex scalar supersymmetric charge $Q^{(+3q)}$,
 - (iii) the $SU(3) \times U(1)$ symmetry of $\mathcal{L}_{3D}^{su_3 \times u_1}$.

We conclude this study by making 2 comments; one concerning the reduction to $2D \mathcal{N} = 4$ dimensions; and the other regarding the extension of the construction to twisted maximal supersymmetric YM in $5D \mathcal{N} = 4$ and $4D \mathcal{N} = 4$ dimensions.

1) *Reduction down to 2D*

The twisted $2D \mathcal{N} = 4$ SYM, that uses the following $SU(3)$ packaging of the fields

$$\begin{array}{lll} \text{bosons} & : & \mathcal{G}^a \quad , \quad \bar{\mathcal{G}}_a, \\ \text{fermions} & : & \xi \quad , \quad \psi^a \end{array}$$

lives on the lattice \mathbb{A}_2^* ; it follows from the $3D \mathcal{N} = 4$ analysis by taking the limit $q = 0$. However, to exhibit the decomposition

$$\begin{array}{ll} \mathbb{Q}_{2 \times 2} & = \quad QI + Q_\mu \gamma^\mu + Q_{\mu\nu} \gamma^{[\mu\nu]} \\ 4 & = \quad 1 + 2 + 1 \end{array}$$

using 2 dimensional gamma matrices γ^μ to decompose $\mathbb{Q}_{2 \times 2}$ in a similar way to the splitting eq(1.1), we have to break the $SU(3) \times U(1)$ symmetry down to

$$SU(2) \times U(1) \times U(1)$$

As a consequence of this breaking, twisted $3D \mathcal{N} = 4$ algebra leads to a particular class of twisted $2D \mathcal{N} = 4$ supersymmetry with generators as follows

$$\begin{array}{ll} SU(3) \times U(1) & \rightarrow \quad SU(2) \times U(1)_{diag} \\ Q^{(+3q)} & Q^{(+3q)} \\ Q_a^{(-q)} & Q_\alpha^{(-p-q)} \quad , \quad Q^{(+2p-q)} \\ P_a^{(+2q)} & P_\alpha^{(-p+2q)} \quad , \quad Z^{(+2p+2q)} \end{array}$$

where $U(1)_{diag}$ is the diagonal subgroup of $U(1) \times U(1)$. This superalgebra has two complex $SU(2)$ scalar supercharges $Q^{(+3q)}$, $Q^{(+2p-q)}$ and an isodoublet $Q_\alpha^{(-p-q)}$ obeying amongst others the anticommutation relations

$$\begin{array}{ll} \{Q^{(+3q)}, Q_\alpha^{(-p-q)}\} & = \quad 2P_\alpha^{(-p+2q)} \\ \{Q^{(+3q)}, Q^{(+2p-q)}\} & = \quad 2Z^{(+2p+2q)} \end{array}$$

where $P_\alpha^{(-p+2q)}$ refers to bosonic translations and where the charge $Z^{(+2p+2q)}$ can be taken equal to zero ($Z^{(+2p+2q)} = 0$) if we want to realize both scalar supersymmetries $Q^{(+3q)}$, $Q^{(+2p-q)}$ on lattice.

The field spectrum describing the on shell degrees of freedom of the twisted 2D $\mathcal{N} = 4$ supersymmetry, that follows from the reduction of the twisted 3D $\mathcal{N} = 4$ SYM is, up to some details, given by

$$\begin{array}{llllll}
SO_E(6) & : & & SU(2) \times U(1)_{diag} & & \\
\mathcal{A}_M & : & \mathcal{G}^{\alpha(-2q+p)} & \bar{\mathcal{G}}_a^{(+2q-p)} & \phi^{(-2q-2p)} & \bar{\phi}^{(+2p+2q)} \\
\Psi^A & : & \psi^{\alpha(+q+p)} & \psi^{(-3q)} & \psi^{(+q-2p)} &
\end{array}$$

The complex bosonic fields

$$\mathcal{G}^{\alpha(-2q+p)}, \quad \phi^{(-2q-2p)}$$

transform respectively in the representation 2_{-2q+p} and 1_{-2q-2p} of $SU(2) \times U(1)_{diag}$. Similarly, the complex fermionic fields

$$\psi^{\alpha(+q+p)}, \quad \psi^{(-3q)}, \quad \psi^{(+q-2p)}$$

transform respectively in 2_{+q+p} , 1_{-3q} and 1_{+q-2p} .

The lattice $\mathcal{L}_{2D}^{su_2 \times u_1}$, on which live the twisted lattice 2D $\mathcal{N} = 4$ theory, follows by the reduction of (8.1) and is given by the fibration

$$\begin{array}{ccc}
\mathcal{L}_{1D}^{u(1)_{diag}} & \rightarrow & \mathcal{L}_{2D}^{su_2 \times u_1} \\
& & \downarrow \\
& & \mathcal{L}_{1D}^{su_2} = \mathbb{A}_1^*
\end{array}$$

where the base sublattice \mathbb{A}_1^* is the weight lattice of $SU(2)$.

2) Extension to 5D $\mathcal{N} = 4$ supersymmetry³

The analysis we have given in this paper extends to the case of twisted 5D $\mathcal{N} = 4$ supersymmetric YM having 16 supercharges. The generators of the underlying twisted 5D $\mathcal{N} = 4$ superalgebra carry charges under $SU(5) \times U(1)$ as follows

$$\begin{array}{llll}
\text{fermionic generators} & : & Q^{(+5q)} & Q_a^{(-3q)} & Q^{[ab](+q)} \\
SU(5) \times U(1) & : & 1_{+5q} & \bar{5}_{-3q} & 10_{+q}
\end{array}$$

³see footnote 2

The field spectrum describing the on shell degrees of freedom of this gauge theory is obtained in 2 steps as follows: first reducing the $\mathcal{N} = 1$ gauge multiplet (\mathcal{A}_M, Ψ^A) in $10D$ down to $5D$ to get

$$(A_\mu, B_m, \Psi^{\alpha I})$$

transforming into representations of

$$SO_E(5) \times SO_R(5)$$

then twisting the two $SO(5)$ factors. In doing these steps, one ends, up on complexification, with the complex field spectrum

$$\begin{array}{lll} SO_E(10) & : & SU(5) \times U(1) \\ \mathcal{A}_M & : & \mathcal{G}^{a(-2q)} \quad \bar{\mathcal{G}}_a^{(+2q)} \\ \Psi^A & : & \psi^{(-5q)} \quad \psi^{a(+q)} \quad \psi_{[ab]}^{(-q)} \end{array}$$

Applying similar techniques used in this paper, one concludes that the $5D$ lattice on which live the twisted $5D$ $\mathcal{N} = 4$ supersymmetric YM should be given by the fibration

$$\begin{array}{ccc} \mathcal{L}_{1D}^{u(1)_{diag}} & \rightarrow & \mathcal{L}_{5D}^{su_5 \times u_1} \\ & & \downarrow \\ & & \mathcal{L}_{4D}^{su_5} = \mathbb{A}_4^* \end{array}$$

with base sublattice \mathbb{A}_4^* precisely as the one found in [1, 3]. More details and special features of this lattice will be reported in a future occasion.

9 Appendix: Building the set covariant superfields

The aim of this appendix is to derive the set (3.20) of the gauge covariant superfields $\Phi_i^{(q_i)}$ for describing twisted chiral $3D$ $\mathcal{N} = 4$ supersymmetric YM theory. A summary of this analysis has been given in subsection 3.2.

9.1 General on scalar supersymmetry in superspace

First recall that the on shell degrees of freedom of the twisted chiral $3D \mathcal{N} = 4$ supersymmetric YM are as follows

$$\begin{array}{lll}
\text{Fermions} & : & \psi^{(-3q)} \quad \psi^{a(+q)} \\
SU(3) \times U(1) & & \bar{1}_{-3q} \quad 3_{+q} \\
\text{scale mass dim} & & 1 \quad 1 \\
\\
\text{Bosons} & : & \mathcal{G}^{a(-2q)} \quad \bar{\mathcal{G}}_a^{(+2q)} \\
SU(3) \times U(1) & & 3_{-2q} \quad \bar{3}_{+2q} \\
\text{scale mass dim} & & \frac{1}{2} \quad \frac{1}{2}
\end{array} \tag{9.1}$$

Using the scalar Grassman variable $\theta^{(-3q)}$, associated with the scalar supersymmetric charge $Q^{(+3q)}$, and auxiliary fields, one may a priori combine these degrees of freedom into particular superfields as follows

$$\begin{aligned}
\Psi^{(-3q)} &= \psi^{(-3q)} + \theta^{(-3q)} \mathbf{F}^{(0)} \\
\mathcal{V}^{a(-2q)} &= \mathcal{G}^{a(-2q)} + \theta^{(-3q)} \psi^{a(+q)} \\
\Upsilon_a^{(-q)} &= \gamma_a^{(-q)} + \theta^{(-3q)} \mathbf{G}_a^{(+2q)}
\end{aligned} \tag{9.2}$$

Notice that the component fields F are not ordinary fields since they depend, in addition to the bosonic coordinates z, \bar{z} , on extra Grassman coordinates $\vartheta^{a(+q)}$ associated with the supersymmetric charges $Q_a^{(-q)}$; that is

$$F = F(z, \vartheta) \tag{9.3}$$

Explicitly, we have

$$\begin{aligned}
\psi^{(-3q)}(z, \vartheta) &= \psi^{(-3q)}(z) + \vartheta^{a(-q)} \xi_a^{(-2q)}(z) + \dots \\
\mathbf{F}^{(0)}(z, \vartheta) &= \mathbf{F}^{(0)}(z) + \vartheta^{a(-q)} \xi_a^{(+q)}(z) + \dots \\
\mathbf{G}^{a(-2q)}(z, \vartheta) &= \mathcal{G}^{a(-2q)}(z) + \vartheta^{b(-q)} \xi_b^{a(-q)}(z) + \dots \\
\psi^{a(+q)}(z, \vartheta) &= \psi^{a(+q)}(z) + \vartheta^{b(-q)} \Delta_b^{a(+2q)}(z) + \dots \\
\gamma_a^{(-q)}(z, \vartheta) &= \gamma_a^{(-q)}(z) + \vartheta^{b(-q)} \xi_{ba}^{(0)}(z) + \dots \\
\mathbf{G}_a^{(+2q)}(z, \vartheta) &= \mathcal{G}_a^{(+2q)}(z) + \vartheta^{b(-q)} \xi_{ba}^{(+3q)}(z) + \dots
\end{aligned} \tag{9.4}$$

The dependence of these component modes into $\vartheta^{a(-q)}$ is eliminated at the end after integration with respect to $\theta^{(-3q)}$ by setting $\vartheta^{a(-q)} = 0$.

Notice moreover that the superfields (9.2) are not good candidates for superspace formulation of scalar supersymmetric invariance. The point is that under gauge symmetry transformations with generic group elements G , the bosonic gauge superfields $U^{a(-2q)}$ and $V_a^{(+2q)}$ do not transform covariantly since

$$\begin{aligned}\mathcal{G}^{a(-2q)} &\rightarrow G\mathcal{G}^{a(-2q)}G^{-1} + G\partial^{a(-2q)}G^{-1} \\ \mathcal{G}_a^{(+2q)} &\rightarrow G\mathcal{G}_a^{(+2q)}G^{-1} + G\partial_a^{(+2q)}G^{-1}\end{aligned}\tag{9.5}$$

To overcome this difficulty, one needs to work with the gauge covariant superfield operators

$$\begin{aligned}\mathcal{D}^{(+3q)} &= D^{(+3q)} + ig_{YM}\Upsilon^{(+3q)} \\ \mathcal{D}_a^{(-q)} &= D_a^{(-q)} + ig_{YM}\Upsilon_a^{(-q)} \\ L_a^{(+2q)} &= \partial_a^{(+2q)} + ig_{YM}V_a^{(+2q)} \\ L^{a(-2q)} &= \partial^{a(-2q)} + ig_{YM}U^{a(-2q)}\end{aligned}\tag{9.6}$$

and their graded commutators from which we learn the set of gauge covariant superfields (3.20); this set is constructed below.

9.2 Gauge covariant superfields

We first give our result regarding the set of gauge covariant superfields; then we turn to derive it explicitly.

9.2.1 the set of superfields

Twisted chiral $3D \mathcal{N} = 4$ supersymmetric YM exhibiting manifestly the supercharge $Q^{(+3q)}$ is described in superspace by the following superfields

$$\begin{array}{lll}
\text{Fermionic sector} & : & \Psi^{(-3q)} \quad \Phi_{ab}^{(+q)} \quad \Psi^{a(+q)} \\
SU(3) \times U(1) & : & \bar{1}_{-3q} \quad 3_{+q} \quad 3_{+q} \\
\text{scale mass dim} & & 1 \quad 1 \quad 1 \\
\\
\text{Bosonic sector} & : & \mathbb{J}^{(0)} \quad \mathbb{E}^{ab(-4q)} \quad \mathbb{F}_{ab}^{(+4q)} \\
SU(3) \times U(1) & : & 1_0 \quad \bar{3}_{-4q} \quad 3_{+4q} \\
\text{scale mass dim} & & \frac{3}{2} \quad \frac{3}{2} \quad \frac{3}{2}
\end{array} \tag{9.7}$$

obeying constraint relations to be derived later on. Their θ -expansion are given by

$$\begin{aligned}
\Psi^{(-3q)} &= \psi^{(-3q)} + \theta^{(-3q)} F^{(0)} \\
\Phi_{ab}^{(+q)} &= \phi_{ab}^{(+q)} + \theta^{(-3q)} \mathcal{F}_{ab}^{(+4q)} \\
\Psi^{a(+q)} &= \psi^{a(+q)} + \theta^{(-3q)} f^{a(+2q)} \\
\\
\mathbb{J}^{(0)} &= \mathcal{J}^{(0)} + \theta^{(-3q)} \nabla_a^{(+2q)} \psi^{a(+q)} \\
\mathbb{E}^{ab(-4q)} &= \mathcal{F}^{ab(-4q)} + \theta^{(-3q)} \left[\nabla^a(-2q) \psi^{b(+q)} - \nabla^b(-2q) \psi^{a(+q)} \right] \\
\mathbb{F}_{ab}^{(+4q)} &= \mathcal{F}_{ab}^{(+4q)} + \theta^{(-3q)} \mathcal{K}_{ab}^{(+7q)}
\end{aligned} \tag{9.8}$$

In these relations $\psi^{(-3q)}$, $\psi^{a(+q)}$ are the twisted fermionic fields of the on shell spectrum (9.1); and $\mathcal{J}^{(0)}$, $\mathcal{F}^{ab(-4q)}$, $\mathcal{F}_{ab}^{(+4q)}$ as follows

$$\begin{aligned}
\mathcal{F}_{ab}^{(+4q)} &= \frac{1}{ig_{YM}} \left[\nabla_a^{(+2q)}, \nabla_b^{(+2q)} \right] \\
\mathcal{E}^{ab(-4q)} &= \frac{1}{ig_{YM}} \left[\nabla^a(-2q), \nabla^b(-2q) \right] \\
\mathcal{J}^{(0)} &= \frac{1}{ig_{YM}} \left[\nabla_a^{(+2q)}, \nabla^a(-2q) \right]
\end{aligned} \tag{9.9}$$

with gauge covariant derivatives as in eq(3.30) and gauge coupling constant g_{YM} scaling like $(mass)^{\frac{1}{2}}$.

9.2.2 Deriving eqs(9.7)

We begin by the superspace realization of the twisted chiral $3D \mathcal{N} = 4$ algebra generated by

$$D^{(+3q)}, \quad D_a^{(-q)}, \quad \partial_a^{(+2q)}, \quad \partial^a(-2q) \quad (9.10)$$

obeying the anticommutation relations

$$\begin{aligned} \left\{ D^{(+3q)}, D_a^{(-q)} \right\} &= 2\partial_a^{(+2q)} \\ \left\{ D_a^{(-q)}, D_b^{(-q)} \right\} &= 0 \\ \left\{ D^{(+3q)}, D^{(+3q)} \right\} &= 0 \end{aligned} \quad (9.11)$$

To implement gauge symmetry, we covariantize the supersymmetric derivatives (9.10) which become

$$\begin{aligned} \mathcal{D}^{(+3q)} &= D^{(+3q)} + i\Upsilon^{(+3q)} \\ \mathcal{D}_a^{(-q)} &= D_a^{(-q)} + i\Upsilon_a^{(-q)} \\ L_a^{(+2q)} &= \partial_a^{(+2q)} + iV_a^{(+2q)} \\ L^a(-2q) &= \partial^a(-2q) + iU^a(-2q) \end{aligned} \quad (9.12)$$

where $\Upsilon_i^{(q_i)}$, $\Upsilon^{(+3q)}$, $V_a^{(+2q)}$, $U^a(-2q)$ are gauge connexions. These superfield operators transform covariantly under arbitrary gauge transformation superfield matrices \mathbf{G} like

$$\begin{aligned} \mathcal{D}^{(+3q)} &\rightarrow \mathbf{G}\mathcal{D}^{(+3q)}\mathbf{G}^{-1} \\ \mathcal{D}_a^{(-q)} &\rightarrow \mathbf{G}\mathcal{D}_a^{(-q)}\mathbf{G}^{-1} \\ L_a^{(+2q)} &\rightarrow \mathbf{G}L_a^{(+2q)}\mathbf{G}^{-1} \\ L^a(-2q) &\rightarrow \mathbf{G}L^a(-2q)\mathbf{G}^{-1} \end{aligned} \quad (9.13)$$

with

$$\mathbf{G} = \mathbf{G}(z, \bar{z}, \vartheta^{a(+q)}; \theta^{(-3q)}) \quad (9.14)$$

which, upon expanding in $\theta^{(-3q)}$ - series, reads also

$$\mathbf{G} = \mathbf{g} + \theta^{(-3q)}\zeta^{(+3q)}$$

with

$$\begin{aligned} \mathbf{g} &= \mathbf{g}(z, \bar{z}, \vartheta^{a(+q)}) \\ \zeta^{(+3q)} &= \zeta^{(+3q)}(z, \bar{z}, \vartheta^{a(+q)}) \end{aligned} \quad (9.15)$$

These gauge covariant derivatives (9.12) are not independent; they obey some constraint relations required by supersymmetry; in particular the conventional ones

$$\begin{aligned}\left\{\mathcal{D}^{(+3q)}, \mathcal{D}_a^{(-q)}\right\} &= 2L_a^{(+2q)} \\ \left\{\mathcal{D}_a^{(-q)}, \mathcal{D}_b^{(-q)}\right\} &= 0 \\ \left\{\mathcal{D}^{(+3q)}, \mathcal{D}^{(+3q)}\right\} &= 0\end{aligned}\tag{9.16}$$

and

$$\left[\mathcal{D}^{(+3q)}, L_a^{(+2q)}\right] = 0\tag{9.17}$$

Being the basic gauge covariant objects of the twisted SYM theory, eqs(9.12) allow to build gauge covariant superfields by taking graded commutators. The gauge covariant superfields with small scaling mass dimension are of particular interest; we have:

1) Bosonic superfields

The bosonic gauge covariant superfields that scale as $(mass)^{\frac{4-1}{2}}$ are given by the commutators of $L_a^{(+2q)}$ and $L^{a(-2q)}$ as follows

$$\begin{aligned}\mathbb{J}^{(0)} &= \frac{1}{ig_{YM}} \left[L_a^{(+2q)}, L^{a(-2q)} \right] \\ \mathbb{E}^{ab(-4q)} &= \frac{1}{ig_{YM}} \left[L^{a(-2q)}, L^{b(-2q)} \right] \\ \mathbb{F}_{ab}^{(+4q)} &= \frac{1}{ig_{YM}} \left[L_a^{(+2q)}, L_b^{(+2q)} \right]\end{aligned}\tag{9.18}$$

with g_{YM} the gauge coupling constant scaling as $(mass)^{\frac{1}{2}}$.

Because of (9.17), the superfield $\mathbb{F}_{ab}^{(+4q)}$ obeys the remarkable property

$$\mathcal{D}^{(+3q)} \mathbb{F}_{ab}^{(+4q)} = 0\tag{9.19}$$

but the two others do not

$$\begin{aligned}\mathcal{D}^{(+3q)} \mathbb{J}^{(0)} &\neq 0 \\ \mathcal{D}^{(+3q)} \mathbb{E}^{ab(-4q)} &\neq 0\end{aligned}\tag{9.20}$$

From these constraint eqs, we learn that $\mathbb{F}_{ab}^{(+4q)}$ should be a highest component of a superfield while $\mathbb{F}_{ab}^{(+4q)}$ and $\mathbb{E}^{ab(-4q)}$ are good candidates for superspace formulation of twisted chiral supersymmetric YM.

2) Fermionic superfields

These fermionic gauge covariant superfields we need scale as $(mass)^{\frac{3-1}{2}}$; and are given by

$$\begin{aligned}\Psi^{(-3q)} &= \frac{1}{ig_{YM}} \left[\mathcal{D}_a^{(-q)}, L^{a(-2q)} \right] \\ \Psi^{a(+q)} &= \frac{1}{ig_{YM}} \left[\mathcal{D}^{(+3q)}, L^{a(-2q)} \right] \\ \Phi_{ab}^{(+q)} &= \frac{1}{ig_{YM}} \left[\mathcal{D}_a^{(-q)}, L_b^{(+2q)} \right]\end{aligned}\tag{9.21}$$

with $\Psi^{a(+q)}$ obeying the property

$$\mathcal{D}^{(+3q)} \Psi^{a(+q)} = 0\tag{9.22}$$

but

$$\begin{aligned}\mathcal{D}^{(+3q)} \Omega^{(-3q)} &\neq 0 \\ \mathcal{D}^{(+3q)} \Phi_{ab}^{(+q)} &\neq 0\end{aligned}\tag{9.23}$$

Here also, we learn that $\Psi^{a(+q)}$ is a highest component of a superfield while $\Psi^{(-3q)}$ and $\Phi_{ab}^{(+q)}$ are good candidates for the superspace formulation of twisted chiral supersymmetric YM.

3) relations between fermionic and bosonic superfields

Using the anticommutation relations of the twisted chiral superalgebra, one finds that the fermionic and bosonic gauge covariant superfields constructed above are not completely independent; they are related through constraint relations; in particular

$$\begin{aligned}\mathcal{D}^{(+3q)} \Psi^{(-3q)} &= 2\mathbb{J}^{(0)} - \mathcal{D}_a^{(-q)} \Psi^{a(+q)} \\ \mathcal{D}^{(+3q)} \Phi_{ab}^{(+q)} &= 2\mathbb{F}_{ab}^{(+4q)} \\ \mathcal{D}^{(+3q)} \mathbb{E}^{ab(-4q)} &= L^{a(-2q)} \Psi^{b(+q)} - L^{b(-2q)} \Psi^{a(+q)} \\ L_b^{(+2q)} \Psi^{(-3q)} &= -L^{a(-2q)} \Phi_{ba}^{(+q)}\end{aligned}\tag{9.24}$$

Acting on the first relation by $\mathcal{D}^{(+3q)}$ and using the identity $\mathcal{D}^{(+3q)} \mathcal{D}^{(+3q)} = 0$, we

get another constraint relation on the $\mathbb{J}^{(0)}$ superfield

$$\mathcal{D}^{(+3q)}\mathbb{J}^{(0)} = L_a^{(+2q)}\Psi^{a(+q)} \quad (9.25)$$

Doing the same thing for the second relation, we end with the constraint relation (9.19).

In what follows, we choose a particular frame for the gauge fields to build the θ -expansions of the superfields

$$\begin{array}{lll} \Psi^{(-3q)} & \Psi^{a(+q)} & \Phi_{ab}^{(+q)} \\ \mathbb{J}^{(0)} & \mathbb{E}^{ab(-4q)} & \mathbb{F}_{ab}^{(+4q)} \end{array} \quad (9.26)$$

solving the constraint relations (9.24-9.25).

B) *Gauge fixing choice*

To make explicit computations in superspace, we start from the supersymmetric gauge covariant derivatives of eqs(9.12); then make the gauge fixing choice

$$\Upsilon^{(+3q)} = 0 \quad (9.27)$$

leading to $\mathcal{D}^{(+3q)} = D^{(+3q)}$ and then

$$\mathcal{D}^{(+3q)} = \frac{\partial}{\partial\theta^{(-3q)}} \quad (9.28)$$

This particular choice also allows to expand (9.20-9.23) as in eq(9.8). To establish this result, notice first that eq(9.27) corresponds to reducing the set of gauge transformations (9.15) down to the subset of superfield matrices \mathbf{G} having no dependence in $\theta^{(-3q)}$, that is

$$D^{(+3q)}\mathbf{G} = 0 \quad (9.29)$$

By substituting back into (9.15), the superfield matrix \mathbf{G} reduces to \mathbf{g} with

$$\mathbf{g} = \mathbf{g}(z, \bar{z}, \vartheta^{a(+)}) \quad (9.30)$$

and the gauge covariant derivatives behaving generally like

$$\begin{aligned}
\mathcal{D}^{(+3q)} &\rightarrow g\mathcal{D}^{(+3q)}g^{-1} \\
\mathcal{D}_a^{(-q)} &\rightarrow g\mathcal{D}_a^{(-q)}g^{-1} \\
L_a^{(+2q)} &\rightarrow gL_a^{(+2q)}g^{-1} \\
L^{a(-2q)} &\rightarrow gL^{a(-2q)}g^{-1}
\end{aligned} \tag{9.31}$$

become

$$\begin{aligned}
\mathcal{D}^{(+3q)} &= \frac{\partial}{\partial\theta^{(-3q)}} \\
\mathcal{D}_a^{(-q)} &= \frac{\partial}{\partial\theta^{a(+q)}} + \theta^{(-3q)}\nabla_a^{(+2q)} \\
L_a^{(+2q)} &= \nabla_a^{(+2q)} \\
L^{a(-2q)} &= \nabla^{a(-2q)} + i\theta^{(-3q)}\lambda^{a(+q)}
\end{aligned} \tag{9.32}$$

with $\nabla_a^{(+2q)}$, $\nabla^{a(-2q)}$ given by (9.6). Substituting these expressions back into eqs(9.24-9.24), one ends with the following θ - expansions

$$\begin{aligned}
\Psi^{(-3q)} &= \psi^{(-3q)} + \theta^{(-3q)}F^{(0)} \\
\mathbb{J}^{(0)} &= \mathcal{J}^{(0)} + \theta^{(-3q)}\nabla_a^{(+2q)}\psi^{a(+q)} \\
\mathbb{E}^{ab(-4q)} &= \mathcal{E}^{ab(-4q)} + \theta^{(-3q)}\left(\nabla^{a(-2q)}\psi^{b(+q)} - \nabla^{b(-2q)}\psi^{a(+q)}\right) \\
\Phi_{ab}^{(+q)} &= \phi_{ab}^{(+q)} + \theta^{(-3q)}\mathcal{F}_{ab}^{(+4q)}
\end{aligned} \tag{9.33}$$

and remarkably $\Psi^{a(+q)}$ has no $\theta^{(-3q)}$ dependence

$$\Psi^{a(+q)} = \psi^{a(+q)} \tag{9.34}$$

with component field modes as in eqs(9.4). We also have the constraint relations

$$\begin{aligned}
\mathcal{D}^{(+3q)}\Psi^{(-3q)} + \mathcal{D}_a^{(-q)}\Psi^{a(+q)} &= 2\mathbb{J}^{(0)} \\
L_b^{(+2q)}\Psi^{(-3q)} &= -L^{a(-2q)}\Phi_{ba}^{(+q)}
\end{aligned} \tag{9.35}$$

The first constraint is solved as

$$\mathbb{J}^{(0)} = \mathcal{J}^{(0)} + \theta^{(-3q)} \nabla_a^{(+2q)} \psi^{a(+q)} \quad (9.36)$$

and the second leads to

$$\nabla_b^{(+2q)} \psi^{(-3q)} = \nabla^{a(-2q)} \phi_{ab}^{(+q)} \quad (9.37)$$

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